A Continuous-Time Mortgage Default Model

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Abstract We propose a new continuous default model to price mortgage-backed securities (MBS) in the presence of defaults and interest-rate sensitive prepayments. The model can be solved using the eigenfunction expansion approach to solve the stochastic PDE. This model is flexible and can be adapted to a constant default rate (CDR) or to the time-dependent Standard Default Assumption (SDA) used in the industry.

Key Words: Mortgage Valuation, Mortgage-Backed Securities, Defaults

1. Introduction

Historically, defaults have represented a much smaller proportion of mortgage terminations than prepayments; however, in recent years, there have been significant increases in the number of defaults especially in the subprime arena. Figure C1 shows how foreclosure rates have rapidly increased from January 2005 until June 2010. Although defaults are not exactly the same as foreclosures, since a borrower who is able to make payments on a house may choose not to do so if his mortgage is under water, they are quite similar. According to (Bhattacharyya, Berliner and Lieber, 2006) "a default is defined as the event where the borrower loses title to the property." (Bond Market Association, 1999) defines foreclosure as a "legal procedure for enforcing payment of a debt by seizing and selling the mortgaged property." While (Goncharov, 2006), (Gorovoy and Linetsky, 2007) and (Kolbe and Zagst, 2009) present continuous prepayment models, none of them have considered models that include defaults.
(Deng, Quigley, and Van Order, 2000) use a competing-risks model between prepayments and defaults where only the event that occurs first is observed and (Kelly, 2009) have default models with a high level of complexity, but do not address the Standard Default Assumption. The proliferation of toxic assets in the mortgage industry has made defaults an essential part of the mortgage equation and has also shown that protecting investors from defaults tends to distort the market. Recently, Congress has been discussing the dissolution of Fannie Mae and Freddie Mac. This could result in an increase in private-label mortgages which are subject to default. Although many have shied away from defaults due to the complexity, rather than ignoring the problem, we introduce a default model with constant default rates and loss severities which can be expanded to become time-dependent.

2. Methods

2.1 Single Stopping-Time Model

We assume a single mortgage will terminate at a stopping time $\tau \wedge T$ where in the simplest case $\tau$ has an exponential distribution. At that time either a prepayment will occur with probability $\frac{h}{h + \delta}$ or a default will occur with probability $\frac{\delta}{h + \delta}$, where $h$ is the prepayment rate and $\delta$ is the default rate. This differs from the competing risks model in (Deng, Quigley, Van Order, 2000) with two stopping times, one for prepayments and one for defaults. We assume a homogeneous pool of mortgages, so that the likelihood of prepayment and default are the same across the pool. By using one stopping time, we can leverage the model of (Goncharov, 2006) without adding too much complexity. (See Figure C2.) The only difference between the prepayments and defaults in our model is that at time $\tau \wedge T$, for a prepayment, the total balance will be returned, whereas with a default, only a proportion of the balance will be recovered, the remainder being lost. The time value of money lost due to the delay in receiving the proceeds while the property is in foreclosure is included in this loss percentage.

2.2 Pricing a Mortgage with Defaults

To price a continuous mortgage with defaults we need to discount the continuous cash flows. These cash flows consist of Scheduled Principal and Interest, Prepayments, and Principal
Recoveries from defaults. The cash flows, decaying over time by prepayments and defaults, must be discounted by the short rate. Let the \( \sigma \)-Algebra \( \mathcal{F}_t \) represent all the information prior to time \( t \).

Let \( B(t) = B_0 \left( \frac{1 - e^{-m(T-t)}}{1 - e^{-mT}} \right) \) be the scheduled balance of a mortgage after time \( t \). For a pool of identical mortgages the cash flow consists of three parts: the scheduled coupon payments, the prepayments, and the recovery amount after a foreclosed property is sold. The scheduled coupon payment is a constant \( c \). The prepayment amount is the prepayment rate \( h_t \) times the remaining balance. And the recovery amount is a proportion of the default rate times the remaining balance. The cash flows are proportional to the actual balance as opposed to the scheduled balance, so in addition to discounting the cash flows by the short rate \( r_t \) we must also account for the decay in the balance due to prepayment and defaults. This can be accomplished by multiplying the cash flows by the factor \( e^{-\int_t^T (r_s + h_s + \delta_s) ds} \). Thus, the price \( M_t \) of a mortgage maturing at time \( T \) with a default rate and loss severity \( S(t) \) is:

\[
M_t = E \left[ \int_t^T (c + h_u B_u + [1 - S(u)] \delta(u) B(u)) e^{-\int_t^u (r_s + h_s + \delta_s) ds} du \mid \mathcal{F}_t \right].
\]  

We can show

\[
M(t) = B(t) - \int_t^T B(u) R(t, u) du + \int_t^T [mB(u) - S(u) \delta(u) B(u)] Q(t, u) du
\]

where \( Q(t, u) = E \left( e^{-\int_t^u (r_s + h_s + \delta_s) ds} \mid \mathcal{F}_t \right) \) and \( R(t, u) = E \left( e^{-\int_t^u (r_s + h_s + \delta_s) ds} r_u \mid \mathcal{F}_t \right) \). From the no-arbitrage rule the pool of mortgages is priced at par at time zero, that is \( M(0) = B(0) \).

Setting \( t = 0 \) and rearranging equation (2) we have an implicit formula for the mortgage rate:

\[
m = \frac{\int_0^T (1 - e^{-m(T-u)}) R(0, u) du}{\int_0^T (1 - e^{-m(T-u)}) Q(0, u) du} + \frac{\int_0^T (1 - e^{-m(T-u)}) S(u) \delta(u) Q(0, u) du}{\int_0^T (1 - e^{-m(T-u)}) Q(0, u) du}.
\]  

### 2.3 Constant Default Rate

For constant default and severity rates equation (3) reduces to:

\[
m = \frac{\int_0^T (1 - e^{-m(T-u)}) R(0, u) du}{\int_0^T (1 - e^{-m(T-u)}) Q(0, u) du} + S \delta
\]  

From equation (4) we can see that the mortgage rate is equal to the average short rate plus the loss premium which depends upon default rate. The functions \( Q(t, u) \) and \( R(t, u) \) can be
reformulated as infinite series (Gorovoy and Linetsky, 2007). Using our model, we can observe how various default assumptions affect the mortgage rate. The simplest model assumes a constant prepayment rate and a constant default rate over the life of a mortgage. Figure C3 shows how the mortgage rates vary as we adjust the default rate and the loss severity assuming a constant prepayment rate. Observe that if the constant default rate (CDR) is 0%, the yield (mortgage rate) remains constant, but if the severity rate is 0%, the yield increases with the default rate because it is simply added to the prepayment rate and the payments are merely accelerated.

2.4 Piecewise Linear Prepayment and Default Rates

If we allow the default rate to be time-dependent, but require the severity rate to be constant, we can rewrite the formula in equation (3) as:

\[
m = \frac{\int_0^T (1 - e^{-m(T-u)}) R(0,u) du + S \int_0^T (1 - e^{-m(T-u)}) \delta(u) Q(0,u) du}{\int_0^T (1 - e^{-m(T-u)}) Q(0,u) du}.
\]

We need to evaluate each of the integrals in equation (5) separately. Let us start with the denominator. First, by substituting the eigenfunction expansion derived in (Gorovoy and Linetsky, 2007) in (2):

\[
Q(0,u) = \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) e^{-\int_0^u h_0(s) + \delta(s) ds - \lambda_n u} \tag{6}
\]

The denominator of equation (5) becomes equation (7) after we embed equation (6) and pull out the summation from the integral:

\[
Q(0,u) = \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) e^{-\int_0^u h_0(s) + \delta(s) ds - \lambda_n u} \tag{7}
\]

Define the time integral

\[
L(\lambda_n, m) = \int_0^T (1 - e^{-m(T-u)}) e^{-\int_0^u h_0(s) + \delta(s) ds - \lambda_n u} \tag{8}
\]

which is identical to the definition in (Gorovoy and Linetsky, 2007) except for the introduction of the default rate. We use the abbreviated form \(L(\lambda_n, m)\) to represent the time integral \(L(0,T; \lambda_n, m)\).

We can now write the denominator of equation (5) as:
\[
\int_{0}^{T} (1 - e^{-m(T-u)})Q(0,u)du = \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0)L(\lambda_n, m) \tag{9}
\]

In a similar fashion
\[
\int_{0}^{T} (1 - e^{-m(T-u)})R(0,u)du = \sum_{n=1}^{\infty} c_n^R \varphi_n(r_0)L(\lambda_n, m) \tag{10}
\]

Now the remaining term, whose coefficient is the severity rate \( S \), can be rewritten as:
\[
\int_{0}^{T} \delta(u)(1 - e^{-m(T-u)})e^{-\int_{0}^{u} h_0(s)+\delta(s)ds} \sum_{n=1}^{\infty} e^{-\lambda_n u} c_n^Q \varphi_n(r_0)du \tag{11}
\]

Pulling out the summation from the integral gives
\[
\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) \int_{0}^{T} \delta(u)(1 - e^{-m(T-u)})e^{-\int_{0}^{u} h_0(s)+\delta(s)ds} e^{-\lambda_n u} \tag{12}
\]

We define a new "default" time integral:
\[
D(\lambda_n, m) = \int_{0}^{T} \delta(u)(1 - e^{-m(T-u)})e^{-\int_{0}^{u} h_0(s)+\delta(s)ds} e^{-\lambda_n u} \tag{13}
\]

so equation (12) can be written as:
\[
\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0)D(\lambda_n, m) \tag{14}
\]

From equations (9), (10) and (14) we can write equation (5) as
\[
m = \sum_{n=1}^{\infty} c_n^R \varphi_n(r_0)L(\lambda_n, m) + S \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0)D(\lambda_n, m) \tag{15}
\]

Rearranging terms of equation (15) gives:
\[
\sum_{n=1}^{\infty} \varphi_n(r_0) \left[ (mc_n^Q - c_n^R) L(\lambda_n, m) - Sc_n^Q D(\lambda_n, m) \right] = 0 \tag{16}
\]

Equation (16) can be solved for \( m \) using an iterative technique such as the Secant Method.

First, however, we must show that the quantities \( L(\lambda_n, m) \) and \( D(\lambda_n, m) \) can be expressed in closed form.

2.5 Industry Prepayment and Default Models

The Public Securities Administration (PSA) and the Standard Default Assumption (SDA) models are piecewise linear deterministic prepayment and default models respectively, used
by (The Bond Market Association, 1999) and discussed by (Fabozzi, Ramsey, Ramirez, 1994). The integrals in the exponents in equations (8) and (13) involve the sum of prepayment and default rates. The sum $h_0(s) + \delta(s)$ is piecewise linear if both $h_0(s)$ and $\delta(s)$ are piecewise linear. The standard prepayment rate using the PSA model increases linearly over the first 30 months to 6% and then remains constant thereafter. (Gorovoy and Linetsky, 2007) extend this to the continuous model:

$$h_0(t) = b \left( at \mathbf{1}_{\{t < T_1\}} + aT_1 \mathbf{1}_{\{t \geq T_1\}} \right)$$

(17)

where $a = 0.024$ and $T_1 = 2.5$.

The parameter $b$ represents the prepayment speed; for 100% PSA, $b = 1$. The Standard Default Assumption (SDA) established by Bond Market Association (1999) applies to 30-year fixed-rate mortgages only. For 100% SDA the default rate starts at zero and rises linearly to 0.6% per annum over 30 months. During the next 30 months it remains constant at 0.6%, then over the next 60 months it drops to 0.03% where it remains constant until the end. 150% SDA would simply multiply the default rate by 1.5. The default rate using the SDA model can be expressed mathematically as:

$$\delta(t) = b_1 \left( a_1 t \mathbf{1}_{\{t < T_1\}} + a_1 T_1 \mathbf{1}_{\{T_1 \leq t \leq T_2\}} - a_2 (t - T_2) \mathbf{1}_{\{T_1 \leq t \leq T_3\}} - a_2 (T_3 - T_2) \mathbf{1}_{\{t \leq T_3\}} \right)$$

(18)

where $b_1$ is the default speed; e.g. for 150% SDA, $b_1 = 1.5$, $a_1 = 0.0024$, $a_2 = 0.00114$, $T_1 = 2.5$ years (30 months), $T_2 = 5$ years ($T_1$ + 30 months) and $T_3 = 10$ years ($T_2$ + 30 months). It is perhaps easier to understand the SDA model by defining it in piecewise linear form:

$$\delta(t) = b_1 \begin{cases} 
  a_1 t, & 0 \leq t \leq T_1 \\
  a_1 T_1, & T_1 \leq t \leq T_2 \\
  a_1 T_1 + a_2 T_2 - a_2 t, & T_2 \leq t \leq T_3 \\
  a_1 T_1 + a_2 T_2 - a_2 T_3, & T_3 \leq t \leq T 
\end{cases}$$

(19)
Substituting numerical values we get the following

\[ \delta(t) = b_1 \begin{cases} 
0.024t, & 0 \leq t \leq 2.5 \\
0.006, & 2.5 \leq t \leq 5 \\
0.0117 - 0.00114t, & 5 < t \leq 10 \\
0.003, & 10 < t \leq 30 
\end{cases} \]

Figure C4 shows the Constant Default Rate (CDR) versus the Standard Default Assumption (SDA) values in equation (18) at varying speeds as set by the Bond Market Association. Figure C5 combines the PSA prepayment model with the Standard Default Assumption to show the overall prepayment rate.

The cumulative default rate using the SDA model is:

\[ \int_0^u \delta(s) dt = b_1 \begin{cases} 
0.0012u^2, & 0 \leq u \leq 2.5 \\
0.006u - 0.0075, & 2.5 \leq u \leq 5 \\
-0.00057u^2 + 0.0117u - 0.02175, & 5 < u \leq 10 \\
0.0003u + 0.03525, & 10 < u \leq 30 
\end{cases} \]  \hspace{1cm} (20)

The PSA rate can be expressed as per gl2007:

\[ \int_0^u h_0(s) ds = b \begin{cases} 
\frac{1}{2}au^2, & u \leq T_1 \\
-\frac{1}{2}aT_1^2 + aT_1u, & u \geq 1 
\end{cases} \]  \hspace{1cm} (21)

Expressed numerically,

\[ \int_0^u h_0(s) ds = b \begin{cases} 
0.012u^2, & u < 2.5 \\
0.06u - 0.075, & u \geq 2.5 
\end{cases} \]  \hspace{1cm} (22)

Combining the default rate (SDA) in equation(20) and the PSA prepayment rate in equation(22) gives:

\[ \int_0^u [h(s) + \delta(s)] ds = \begin{cases} 
0.012b + 0.0012b_1, & 0 \leq u \leq 2.5 \\
0.06bu - 0.006b_1u - 0.075b - 0.0075b_1u, & 2.5 \leq u \leq 5 \\
-0.00057b_1u^2 + (0.0117b_1 + 0.06b)u - (0.02175b_1 + 0.075b), & 5 < u \leq 10 \\
(0.0003b_1 + 0.006b)u + 0.03525b_1 - 0.075b, & 10 < u \leq 30 
\end{cases} \]  \hspace{1cm} (23)
The general form is

\[
\int_0^t [h(s) + \delta(s)] ds = \begin{cases} 
\frac{1}{2} H_1 u^2, & u \leq T_1 \\
-\frac{1}{2} H_1 T_1^2 + H_1 T_1 u, & T_1 \leq u \leq T_2 \\
-\frac{1}{2} (H_1 T_1^2 + H_2 T_2^2) + (H_1 T_1 + H_2 T_2) u - \frac{1}{2} H_2 u^2, & T_2 \leq u \leq T_3 \\
-\frac{1}{2} [H_1 T_1^2 + H_2 (T_2^2 - T_3^2)] + [H_1 T_1 + H_2 (T_2 - T_3)] u, & T_3 \leq u \leq T 
\end{cases}
\] (24)

Using equation (24) we can convert equation (8) to closed form for the combined PSA, SDA model:

\[
L_{PSA,SDA}(\lambda_n, m) = \int_0^{T_1} (1 - e^{-m(T-u)}) e^{\frac{1}{2} H_1 u^2 - \lambda_n u} du + \\
\int_{T_1}^{T_2} (1 - e^{-m(T-u)}) e^{\frac{1}{2} H_1 T_1^2 + H_1 T_1 u - \lambda_n u} du + \\
\int_{T_2}^{T_3} (1 - e^{-m(T-u)}) e^{\frac{1}{2} (H_1 T_1^2 + H_2 T_2^2) + (H_1 T_1 + H_2 T_2) u + \frac{1}{2} H_2 u^2 - \lambda_n u} du + \\
\int_{T_3}^{T} (1 - e^{-m(T-u)}) e^{\frac{1}{2} [H_1 T_1^2 + H_2 (T_2^2 - T_3^2)] + [H_1 T_1 + H_2 (T_2 - T_3)] u - \lambda_n u} du
\] (25)

Equation (25) can be reduced to a sum of integrals in one of the three forms below which can be solved analytically:

\[
\int e^{c_0 + c_1 u} du = -\frac{e^{c_0 + c_1 u}}{c_1} \\
\int e^{c_0 + c_1 u + c_2 u^2} du = -\frac{1}{2} \sqrt{\pi} \exp\left(-\frac{4c_0 c_2 - c_1^2}{4c_2}\right) \text{erf}\left(\frac{2c_2 u + c_1}{2\sqrt{c_2}}\right) \\
\int u e^{c_0 + c_1 u + c_2 u^2} du = -\frac{c_0 + c_1 u + c_2 u^2}{2c_2} - \frac{c_1 \sqrt{\pi} \exp\left(\frac{c_1^2}{4c_2}\right) \text{erf}\left(u \sqrt{c_2} + \frac{c_1}{2\sqrt{c_2}}\right)}{4c_2^{3/2}}
\] (26)

Although the piecewise linear model is extremely complex, its closed form makes it tractable and the mortgage rate can be calculated directly.

3. Results

Although agency MBS are not subject to defaults, the default model in this paper can be useful in several ways. We examine two specific ways to study the pricing of GNMA MBS by
comparing prices and yields of similar instruments both with and without defaults. During periods of economic stress (when the yield curve is humped or inverted) the spread to treasuries tends to increase for MBS; however, by pricing a theoretical GNMA MBS with defaults using the typical spread, we are able to approximate the same price as we did using the crisis spread. Since agency mortgages are guaranteed against defaults, we can use the default model to estimate the value of this guarantee by comparing the yield of an MBS with no defaults with that of a theoretical MBS with defaults. The difference in basis points can then be compared to the guarantee fee charged by Ginnie Mae, for example.

3.1 Using Defaults in Lieu of a Crisis Spread

When the yield curve is not in the familiar normal upward-sloping form, it may be difficult to calibrate interest rate parameters. The interest rate models we are using assume that any point in time the long-term interest rate is constant. More complex models assume that the interest rate is a deterministic function of time. The simpler parameterization is necessary to allow us to use the eigenfunction expansion method in our pricing model. The downside of this is the difficulty in handling inverted and humped yield curves. Sometimes $κ$ becomes extremely large. Most of the time $θ$ is fairly stable, but occasionally even this parameter may become unusually large. When this occurs, we need to look at an alternative method of calibration. Many cases where the yield curve does not calibrate well are in times of economic crises. Since we use Single A Corporate Bond rates as a benchmark for Mortgage-Backed Securities, we observe the spread between them and U.S. Treasuries. This spread tends to increase during periods of economic stress because of additional credit risk. One example is in 2000 during the Internet bubble. The spread also went up between 2005 and the financial crisis of 2008. Since the Bloomberg Corporate A yields are not available before October 2005, we can estimate the crisis spread by looking at the period between 2005 and 2008. (After October 2008, the yields are too unstable.) Our estimate for the crisis spread is 95.4 basis points. Since we were unable to calibrate Treasury rates plus the normal spread during the crisis period, we will use the parameters from the calibration of the crisis spread of 95.4 basis points. However, since the difference between the crisis spread and the normal spread is 21.4 basis points, we will adjust the long term rate parameter by this amount and
use the adjusted parameter in our default model: \( \tilde{\theta} = \theta - 0.214 \).

We now apply the default model to GNMA 6.0%, 6.5%, 7.0%, 7.5% and 8.0% MBS. Figure C shows the default model prices versus the crisis spread prices along with market prices from Bloomberg and model prices from (Kolbe Zagst, 2009). Observe that the Default Models and Crisis Spread Models are very close to each other over this period. There is a small spread for the 8.0% GNMA, but both models are better than Kolbe-Zagst.

### 3.2 Determining the Value of the GNMA Guarantee Fee

For a six-basis point guarantee fee on the balance of the securities, GNMA pays the issuer the lost principal and interest from defaults each month. Although defaulting mortgages take up a year to be settled through the foreclosure process, we treat a default as an immediate prepayment except that only a fraction of the total balance due is received at that time. The proportion of the total balance due is based on three factors—the recovery amount, the current mortgage rate and the time to foreclosure. The recovery amount is the market value of the house, less foreclosure expenses incurred by the bank. Since many mortgages are currently underwater, this recovery amount should be substantially less than it was prior to 2008. To account for the delay in receiving the recovery amount, we must also discount it by the mortgage rate for the period from first default to foreclosure.

What is the value of the guarantee provided by Ginnie MAE? To determine this, we must first determine the yield of an MBS with prepayments and zero defaults. We then determine the yield of an MBS with prepayments, a specified default rate and a loss severity rate. The difference between the yields determines the minimum number of basis points necessary for the guarantee fee to cover the losses associated with default.

In a theoretical example from (Gorovoy and Linetsky, 2007), the short rate is \( r_0 = 0.09, \theta = 0.06, \sigma = 0.10, \kappa = 0.25, k = 0.09 \), and \( \gamma = 5.0 \). Using the constant prepayment and constant default model we assume 4.5% CPR (Constant Prepayment Rate) and 0.6% ADR (Annual Default Rate) respectively. First we ignore the default rate and estimate the mortgage rate for a 30-year mortgage which is 7.85272%. This is the same value obtained in Gorovoy and Linetskys example. Now let us apply a constant default rate (CDR) of 0.6% and loss severity of 20%. This results in a mortgage rate of 8.01942%. The difference between the two rates
is 16.65 basis points. This assumes however, that the default rate of 0.6% remains constant throughout the life of the pool.

A slightly more realistic approach is to use the PSA prepayment model with the standard default assumption (SDA) described in Section 2.5 where the prepayment and default rates depend upon the seasoning of the pool. For illustration purposes, we take another example using CIR parameters from previously calibrated by (Gorovoy and Linetsky, 2007):

$$\theta = 0.0702, \kappa = 0.147, \sigma = 0.095, r_0 = 0.0395, b = 0.545, \gamma = 15.5, s = 0.00375.$$ For a zero default rate, the interest rate is 5.61223%.

Table B1 shows what happens to the mortgage rate by varying the default rate as a multiple of SDA and varying the loss severity from 0% to 50%. The table also shows the difference from the no-default interest rate produces the following table of loss premiums. Observe that the loss premium is negative for very low (1%) loss-severity rates. This is because the yield curve predicts that interest rates will rise and the losses are too small to offset this advantage. To eliminate this ”negative” loss premium, we can compare the yields to the same default rate with a loss severity of 0. These differences are also listed in Table B1. Here the loss premiums up to 100% SDA are small enough for the 6 basis-point guarantee fee to be effective. However, in recent years, foreclosure rates and hence default rates have increased substantially. Furthermore, an increase in loss severity due to the bursting of the recent housing bubble shows that Ginnie Maes guarantee fee is inadequate.

Now let us apply this method to actual data from Ginnie Mae. First we look at a 6.0% GNMA sold on January 31, 2005. The price according to Bloomberg on that date was $P_t = 103.797$. The WAM (weighted average maturity) was 26.5 years. We use the calibrated CIR parameters $\kappa = 0.32368, \sigma = 0.17805$ and $\theta = 0.06210$, the short rate 2.51% and our prepayment estimates of $\gamma = 5.647, h_0 = 0.051$, and prepayment threshold of 5.44%. Solving equation (4) numerically with $\delta = 0$ gives us a baseline mortgage rate of 6.14778%.

(This is approximately the same as the effective annual rate of a nominal 6.0% mortgage compounded monthly: 6.168%.) Now let us use the foreclosure rate of 0.95% for January 2005 as a proxy for the default rate. Since this is a constant rate, we use the simpler CPR model with a constant default rate. Let us assume a loss severity of 20%. Using the function $S^*(m) = 1 - (1 - 0.2)e^{-m}$ from above we find that the adjusted loss severity $S^* = 0.246476$. From this we find that the mortgage rate $m = 6.38851\%$. The difference between the mortgage
rate with defaults and the baseline mortgage rate is 24.07 basis points which is very close to the loss premium $S\delta = 0.246476 \times 0.95 = 0.234152$. It is also close to the difference between the normal (74 BP) and crisis (95 BP) spreads we examined in the previous section.

Traditionally a loss severity of 20% was typical for the industry. However, in recent years due to the housing bubble, this amount could easily be much higher. So it is useful to observe the sensitivity of MBS yields to loss severity. Holding everything else constant, we vary the loss severity from 0% to 50% and observe the difference in basis points. Table B2 shows the implied value of the Ginnie Mae Loan guarantee by comparing implied yields of a GNMA 6.0% security at the Bloomberg price of $103.797 on January 31, 2005 with various loss severities to the yield of a no-default GNMA 6.0% MBS and the yield of a GNMA 6.0% with an implied default rate of 0.95% (reflecting the foreclosure rate in January 2005) with no loss severity. The latter yield basically adds the default rate to the prepayment rate. Since the prepayment rate is imbedded in the MBS yield anyway, this implied value may be more accurate.

We define the loss premium as the value of the Ginnie Mae Guarantee. Observe that even in 2005 the loss premium exceeds the six basis points charged by Ginnie Mae. This indicates that the guarantee fee charged by Ginnie Mae is a good deal for most banks and is therefore subsidized.

For a more recent case, we look at a GNMA 5.5% MBS sold on July 31st, 2010. The market price from Bloomberg is $108.703; assuming no defaults and solving for the yield, we obtain 5.59669%. The foreclosure rate for July 2010 was extremely high at 11.1%; however, at this rate nearly two-thirds of all mortgages will have defaulted after 10 years. Instead we will average the foreclosure rates over the previous four years to approximate the annual default rate (ADR); this results in a default rate of 6.664%. The implied value of the loan guarantee at this default rate for the July 2010 MBS is also listed in Table B2. We also include loss severities of 40% and 50%, since housing prices fell substantially after 2008. It is clear from this table that 6 basis points are inadequate to cover the losses from defaults even if we assume a typical 20% loss severity.
4. Conclusion

Although GSEs such as Fannie Mae, Freddie Mac and Ginnie Mae protect investors in their securities from defaults by guaranteeing the payments; we can use the default model to evaluate Agency Mortgage Backed Securities indirectly. The loss premium added by the default model to the mortgage rate can be used to account for the larger spread to treasuries during crisis periods. Furthermore we can estimate the spread between the yield of an MBS with defaults and the yield or an MBS without defaults to determine the intrinsic value of the mortgage guarantee provided by these agencies.

Appendix A. References

References


Appendix B. Tables

Appendix C. Figures
Sensitivity of Mortgage Rates to Standard Default Assumption and Loss Severity

<table>
<thead>
<tr>
<th>% SDA</th>
<th>0%</th>
<th>1%</th>
<th>10%</th>
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<th>30%</th>
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Basis Point difference to 0% Loss Severity and 0% SDA

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<th>10%</th>
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Basis point Difference to 0% Loss Severity with Same Default Rate

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Table B1. Effect of Defaults on Mortgage Rates (SDA Model)

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<tr>
<th>MBS (Seasoned)</th>
<th>Loss Severity</th>
<th>CPR/CDR Required Yield (%)</th>
<th>Implied Value of Loan Guarantee (BP)</th>
<th>$\delta = 0, S = 0$</th>
<th>$S = 0$</th>
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<td>GNMA</td>
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<tr>
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<td>WAM (yrs)</td>
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Table B2. Sensitivity of Implied Loan Guarantee Values to Loss Severity
Figure C1. Foreclosure rates from Jan 2005 - June 2010 (Source Bloomberg)

Figure C2. Single Stopping-Time Model
Figure C3. 30-year Mortgage Rates under varying default and loss assumptions holding exogenous prepayment rate constant at 4.5%
Figure C4. Comparison of constant Default Rate (CDR) to Standard Default Assumption (SDA)

Figure C5. Industry Prepayment and Default Models
Figure C6. GNMA Default Model vs. Crisis Spread for Various Coupons