Miller-Orr Model Revisited

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Abstract

Miller-Orr model is used to optimize the cash management at a corporation. Based on the premise that cash flows are stochastic in nature, the firm is minimizing its overall cost of having cash and marketable securities. The present paper fills the gaps in the development and understanding of this model. In particular, it calculates the optimal level of transfer costs and the interest forgone. Finally, Monte Carlo calculations verify the validity of this model.

1. Introduction

William Baumol [Baumol, 1952] presented the first analytical treatment of the cash management of a firm. He treated the cash balance in a checking account as an inventory item. This simple model assumes that the firm maintains two separate accounts: the brokerage account that houses the marketable securities and the checking account, which is used to pay all the expenses. In 1952, according to federal regulations, the banks did not pay interest on checking accounts and they did not deal with marketable securities.

Figure 1: Cash flow in the Baumol model of cash management. The company maintains a checking account and a brokerage account. The company deposits all incoming cash in a brokerage account, which is invested in high-grade bonds and Treasury securities. The company transfers $x$ dollars at regular intervals from the brokerage account to the checking account. It writes checks to pay all the bills.

Figure 1 shows the arrangement of the two accounts. The firm deposits all cash receipts in the brokerage account and buys marketable securities with this cash. It uses the checking account to pay the expenses. The firm uses the cash until the balance in the checking account is zero and then it replenishes cash by selling enough marketable securities. The model considers two costs, namely, the cost of transferring the money from the brokerage account to the checking account and the implicit cost of interest forgone. It then attempts to minimize the sum of these costs.
The Baumol model assumes that the firm spends its cash uniformly with time. The expenses are entirely predictable and the firm can transfer the cash from the brokerage account to the checking account instantly. Using the concept of economic order quantity, the model develops the well-known result,

\[ x = \sqrt{\frac{2bC}{r}} \]  

In equation (1), \( b \) is the cost of transferring money from brokerage to checking account for each transfer, \( C \) is the total annual disbursements from the checking account, and \( r \) is the annual rate of interest available in the brokerage account. Figure 2 illustrates the cash flows in the checking account.

![Cash Flows Illustration](image_url)

**Figure 2:** An example of a firm that pays out $10,000 uniformly every week in bills. It starts with $10,000 in the checking account, and when the balance drops to zero, it replenishes the cash by another deposit of $10,000. The average amount in the checking account is thus $5000.

Baumol model is an instructive example of cash management in finance pedagogy. However, it is based on unrealistic assumptions. The main objection to this model is its inability to take into account the random nature of cash flows.

2. Miller-Orr Model

To address the shortcomings of Baumol model, Miller and Orr [Miller and Orr, 1966] developed a more realistic model in 1966 by assuming that the cash flows are random. This is a substantial improvement over the previous model. In 1968, they published a second paper to extend the results developed in the first paper [Miller and Orr, 1968] including three assets. Their papers are missing some of the essential details that the students need to understand the model well.

Miller-Orr model makes the following assumptions.

(1) A company is managing its cash by using two accounts. It maintains a checking account, which receives all incoming cash. All the bills are paid out of this account. The second account is an investment account holding the marketable securities, which earn interest at the annual rate \( r \). The optimal amount of the cash in the checking account is \( x \) and the minimum amount is zero. When the balance in the checking account reaches an upper limit, \( h \), then the excess cash \( (h - x) \) is transferred into the investment account. When the balance in the checking account is zero, money is transferred from the investment account, bringing the balance back to \( x \).

Figure 3 depicts the cash flows in the accounts.

![Cash Flow Diagram](image)

Figure 3 shows the use of two accounts in the Miller-Orr cash management system. Money goes in and out of the checking account. If the balance reaches a maximum value \( h \), an amount \( 2x \) is siphoned off to the brokerage account. If the balance reaches a minimum, an amount \( x \) is transferred from the brokerage account to bring it back to its optimal level.

(2) Money is transferred between the two accounts at any time at a given marginal cost of \( b \) per transfer, independent of the size of the transfer, the direction of the transfer or of the time since the previous transfer.

(3) Transfers take place instantaneously, and the "lead-time" for transfers between the two accounts, is negligible.

(4) As a safety measure, the company keeps a minimum amount \( L \) in the checking account, rather than letting it drop to zero. In the current discussion, we will assume that \( L = 0 \).

(5) The most important assumption is that the cash flows are stochastic. The balance in the checking moves up or down \( n \) times each year, where \( n \) is a large number. The change in the balance is \( m \) dollars each time. The cash flow has equal probability of being negative or positive.

The mean change in the cash balance is
\[ \mu = n[\frac{1}{2} m + \frac{1}{2} (-m)] = 0 \]  

(2)

The variance of these changes is

\[ \sigma^2 = \frac{1}{2} (m - 0)^2 + \frac{1}{2} (-m - 0)^2 = m^2 \]  

(3)

This gives the mean, \( \mu = 0 \) and the variance, \( \sigma^2 = m^2 \).

Suppose the lower limit in the checking account is zero. To start the process, transfer \( x \) dollars from the brokerage account to the checking account. When the balance in the checking account reaches \( h \), then we transfer \( h - x \) dollars from the checking account to the brokerage account.

The problem is to find the optimal balance of the checking account \( x \), and the maximum balance \( h \), so that the total cost of the money-management system is minimized. The known quantities in the system are the cost of transferring money between the account for each transfer \( b \), the rate of forgone interest \( r \), and standard deviation of the cash flows \( \sigma^2 \). The unknown quantities are \( x \) and \( h \). Let us write,

Total cost = transactions cost + interest forgone  

(4)

Since this is a statistical problem, we can minimize the expected value of quantities in the previous equation. In symbols, it becomes,

\[ \mathbb{E}(C) = b \mathbb{E}(N) + r \mathbb{E}(B) \]  

(5)

\( \mathbb{E}(C) \) = expected total cost of the cash management system  
\( \mathbb{E}(N) \) = expected number of transfers per year  
\( \mathbb{E}(B) \) = average balance in the checking account

Using statistical theory, we need to evaluate the two quantities, \( \mathbb{E}(N) \) and \( \mathbb{E}(B) \), separately. Then we use calculus to minimize their total expected cost, \( \mathbb{E}(C) \).

The expected number of transfers per year is equal to the number of times the balance reaches 0 or \( h \). To find the answer, we explore random walk theory. An excellent discussion of this topic is available in Steele [Steele, 2001]. Consider the problem of a gambler, who is betting one dollar on the toss of a coin. For heads, he wins a dollar and for tails, he loses a dollar. Suppose he is flipping the coin every second. The game will end when the gambler wins \( A \) dollars or loses \( B \) dollars. The problem is to find the expected duration of the game in seconds. Using elegant mathematical arguments, Steele obtains the surprisingly simple result as equation (1.10) on page 4 of his book [Steele, 2001].

\[ \mathbb{E}(t) = AB \]  

(6)

What happens if the gambler bets \( m \) dollars at each step? The game will end much sooner than \( AB \). Consider it a diffusion problem. The speed of diffusion of winnings or losses is
proportional to $\sigma^2$ of the amount of bets and the gambler will reach the upper or lower limit $\sigma^2$ times sooner. From equation (3), we know that $\sigma^2 = m^2$. This means that the expected time, when the game will be over, is given by

$$E(t) = \frac{AB}{m^2}$$

(7)

Now let us consider a firm managing its cash. It starts with $x$ dollars in the checking account. If the cash reaches an upper level $h$, or a lower level $0$, the firm makes a transfer of cash to bring the balance back to $x$. This is equivalent to $A = h - x$ and $B = x$ in the previous example. Substituting these values of $A$ and $B$ in (7), we find the expected time, $E(t)$, in years, between two cash transfers, as

$$E(t) = \frac{x(h-x)}{m^2}$$

(8)

From equation (3), we know that $m^2 = \sigma^2$. The reciprocal of this quantity gives the expected number of transfers per year,

$$E(N) = \frac{\sigma^2}{x(h-x)}$$

(9)

Now we look at the expected balance in the checking account. The firm starts with a balance of $x$ dollars. The balance jumps up, or down, by an amount $m$ in a random fashion. When the balance reaches a maximum level $h$, it immediately drops back to $x$. Similarly, when the balance becomes 0, it jumps back to $x$. The most suitable probability distribution that applies to the situation is the triangular probability distribution [Forbes, 2011].

A random variable $X$ has the triangular distribution with vertex at $c$ if $X$ has the probability density function $g(x)$ described as follows:

$$g(x) = \begin{cases} 
0, & x < a \\
\frac{2(x - a)}{(b - a)(c - a)}, & a \leq x \leq c \\
\frac{2(b - x)}{(b - a)(b - c)}, & c \leq x \leq b \\
0, & b < x 
\end{cases}$$

(10)

Figure 4 shows the triangle distribution. The distribution has the advantage that it is simple. It relies on a maximum point $b$, and minimum point $a$, and a peak point $c$. It suits well for a company that is managing its cash in an optimal way: there a minimum value of cash in the checking account, 0, a maximum point $h$, and a point $x$, where the probability density is maximum.
The area of the triangle is 1, which represents the total probability. We can calculate the total area as follows.

\[
\int_{a}^{c} \frac{2(x-a)}{(b-a)(c-a)} \, dx + \int_{c}^{b} \frac{2(b-x)}{(b-a)(b-c)} \, dx = 1 \tag{11}
\]

One can establish the above result by direct integration.

The expected value of \(X\), the balance in the checking account, is given by

\[
\mathbb{E}(X) = \int_{a}^{c} \frac{2(x-a)x}{(b-a)(c-a)} \, dx + \int_{c}^{b} \frac{2(b-x)x}{(b-a)(b-c)} \, dx = \frac{a + b + c}{3} \tag{12}
\]

For the firm managing its cash, the minimum balance in the checking account is \(a = 0\), the maximum balance is \(b = h\), and the peak point is \(c = x\). Substituting these values in (12), we get the average balance in the checking account as

\[
\mathbb{E}(B) = \frac{h + x}{3} \tag{13}
\]

Adding the two terms in equation (5), we get the expected cost of the cash management system as

\[
\mathbb{E}(C) = \frac{b\sigma^2}{x(h-x)} + \frac{r(h+x)}{3} \tag{14}
\]
Equation (14) has two unknown variables, $x$ and $h$. To minimize $\mathbb{E}(C)$, we differentiate the right hand side of (14) with respect to $x$ and $h$, and set the derivatives equal to zero. Next, solve the resulting equations for $x$ and $h$. We get

$$\frac{\partial \mathbb{E}(C)}{\partial x} = \frac{-b\sigma^2}{x^2(h-x)} + \frac{b\sigma^2}{x(h-x)^2} + \frac{r}{3} = 0 \tag{15}$$

$$\frac{\partial \mathbb{E}(C)}{\partial h} = \frac{-b\sigma^2}{x(h-x)^2} + \frac{r}{3} = 0 \tag{16}$$

Solving these algebraic equations, we obtain the results for $x$ and $h$:

$$x = \left(\frac{3b\sigma^2}{4r}\right)^{1/5} \tag{17}$$

and

$$h = 3x \tag{18}$$

Equations (17) and (18) represent the essence of the Miller-Orr model. Let us investigate the second order conditions.

$$\frac{\partial^2 \mathbb{E}(C)}{\partial x^2} = \frac{2b\sigma^2(h^2 - 3hx + 3x^2)}{x^3(h-x)^3} \tag{19}$$

When $h = 3x$, it becomes

$$\frac{\partial^2 \mathbb{E}(C)}{\partial x^2} \bigg|_{h=3x} = \frac{3b\sigma^2}{4x^3} > 0 \tag{20}$$

Further,

$$\frac{\partial^2 \mathbb{E}(C)}{\partial h^2} = \frac{2b\sigma^2}{x(h-x)^3} \tag{21}$$

and

$$\frac{\partial^2 \mathbb{E}(C)}{\partial h^2} \bigg|_{h=3x} = \frac{b\sigma^2}{4x^4} > 0 \tag{22}$$

Further,

$$\frac{\partial^2 \mathbb{E}(C)}{\partial h \partial x} = \frac{\partial^2 \mathbb{E}(C)}{\partial x \partial h} = \frac{b\sigma^2(h-3x)}{x^3(h-x)^3} \tag{23}$$

At $h = 3x$, equation (23) gives

$$\frac{b\sigma^2(h-3x)}{x^2(h-x)^2} \bigg|_{h=3x} = 0 \tag{24}$$

From (20), (22) and (24), it becomes apparent that expected total cost, $\mathbb{E}(C)$ has reached a minimum.
3. Separating the costs

Substituting the optimal values of \(x\) and \(h\), it is possible to calculate the transactions cost, interest forgone, and the total cost as follows.

Transactions cost \(= \frac{b\sigma^2}{x(h-x)} = \frac{(6br^2\sigma^2)^{1/3}}{3}\)  \hspace{2cm} (25)

Interest forgone \(= \frac{r(h+x)}{3} = \frac{2(6br^2\sigma^2)^{1/3}}{3}\)  \hspace{2cm} (26)

Total cost \(= \frac{(6br^2\sigma^2)^{1/3}}{3} + \frac{2(6br^2\sigma^2)^{1/3}}{3} = (6br^2\sigma^2)^{1/3}\)  \hspace{2cm} (27)

In Baumol Model, the transactions cost is equal to the interest forgone, whereas in Miller-Orr Model, the interest forgone is twice the transactions cost.

4. Dimensional Analysis

It is instructive to analyze the results of this discussion using dimensional analysis. In finance, all quantities have just two fundamental dimensions: time and money. One could measure time in years, and use \(T\) as its symbol. Similarly, dollars can measure the amount of money, using the symbol $\$. Of course, we have other measures of time and money. We can measure time in days or seconds, and measure money in euros or yens.

In equations (17) and (27), the dimensions of various quantities are as shown in Table 1.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Dimension</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transaction cost, (b)</td>
<td>$</td>
</tr>
<tr>
<td>Interest rate, (r)</td>
<td>per year, or (T^{-1})</td>
</tr>
<tr>
<td>Variance of cash flows, (\sigma^2)</td>
<td>$^2\text{ per year, or } $^2T^{-1}$</td>
</tr>
<tr>
<td>Transfer amount, (x = \left(\frac{3b\sigma^2}{4r}\right)^{1/3})</td>
<td>(\left(\frac{$^2T^{-1}}{T^{-1}}\right)^{1/3} = $)</td>
</tr>
<tr>
<td>Cost per year (= (6br^2\sigma^2)^{1/3})</td>
<td>($T^{-2}$^2T^{-1})^{1/3} = $T^{-1}$</td>
</tr>
</tbody>
</table>

Table 1: The table shows the dimension of various financial quantities, using the dimension of money, $\$, and dimension of time, \(T\).

This implies that the transfer amount is in dollars and the total annual cost is in dollars per year. This shows that the equations are all dimensionally correct.

It is important to note that if the variance is calculated using daily cash inflows and outflows, then we must also use the daily interest rate in the investment account.
5. Monte Carlo Experiment

Since the Miller-Orr model is a statistical model, it is possible to verify its principal findings by using Monte Carlo calculations. Using a random number generator, a Maple plot shows the simulated behavior of random cash flows in a firm. The firm is using Miller-Orr model to manage its cash. The results of this simulation are presented in Figure 5.

![Monte Carlo Simulation](image)

Figure 5 shows the random fluctuations in the cash balance of an idealized firm. The firm maintains a minimum balance of $100 in the account. In this case, \( x = 500 \) and \( h = 1500 \). Starting at the optimal level \( \$600 \), there were 5000 random movements in the balance, each one \( \pm \$30 \). The resulting balance reached a maximum of \( \$1600 \) three times, while it touched the minimum of \( \$100 \) four times during this time interval. Whenever the balance reached a maximum or a minimum, it instantly returned to its optimal value, \( \$600 \).

6. Conclusions

The present paper separates the two costs, the interest forgone and the transfer costs, and presents their values explicitly. Unlike the Baumol model, for Miller-Orr model, the interest forgone is twice the transactions cost. A Monte Carlo simulation verifies the main results of the calculations.

7. References


