Two-Point Distortion Bounds for Biholomorphic Mappings of the Ball in $\mathbb{C}^n$

Jerry R. Muir, Jr.†

Abstract
Lower and upper two-point distortion bounds for families $\mathcal{F}$ of biholomorphic mappings on the unit ball $B$ of $\mathbb{C}^n$ are given in terms of the (trace) order of the linear-invariant family generated by $\mathcal{F}$, bounds on ratios involving the derivative and Jacobian of the mappings in $\mathcal{F}$, and the Carathéodory distance on $B$. (By two-point distortion bounds, we mean estimates on $\|f(b) - f(a)\|$ for $a, b \in B$ and $f \in \mathcal{F}$.) This immediately results in growth bounds for such mappings. A contrast is drawn between these bounds and two-point distortion bounds in terms of the norm order of the generated linear-invariant family. As part of our work, we develop a lower distortion bound for automorphisms of $B$ that may be of independent interest.

1 Introduction

The pursuit of two-point distortion bounds for univalent mappings of the open unit disk $D$ in $\mathbb{C}$ goes back to the work of Blatter [1]. Inspired by his work, Kim and Minda [11] and Ma and Minda [13] provided families of hyperbolically invariant two-point lower and upper distortion bounds for univalent mappings on $D$. We take note, in particular, of the lower bound [11]

$$|f(a) - f(b)| \geq \frac{\sinh 2d_h(a, b)}{2 \exp 2d_h(a, b)} \max\{|f'(a)|(1 - |a|^2), |f'(b)|(1 - |b|^2)|$$

and upper bound [13]

$$|f(a) - f(b)| \leq \frac{\exp 2d_h(a, b) \sinh 2d_h(a, b)}{2} \min\{|f'(a)|(1 - |a|^2), |f'(b)|(1 - |b|^2)|$$

where $f : D \to \mathbb{C}$ is univalent, $a, b \in D$, and $d_h$ is the hyperbolic distance on $D$. There is an appealing symmetry to these bounds that was recently exploited by Hamada, Kohr, and the author in their work on extension operators [9]. It is desirable to have similar bounds for biholomorphic mappings of the open unit ball $B$ of $\mathbb{C}^n$, $n \geq 2$.

As is to be expected, there are obstacles to a simple generalization of these bounds to higher dimensions, the most notable of which is the lack of an analog to the Bieberbach coefficient bound $|a_2(f)| \leq 2$. (Here, $f : D \to \mathbb{C}$ is normalized by $f(0) = 0$, $f'(0) = 1$, and

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†Department of Mathematics, The University of Scranton, Scranton, PA 18510
$a_2(f)$ is the coefficient of $z^2$ in its Taylor series at the origin.) For this reason, our work will involve linear-invariant families of biholomorphic mappings, a concept originated by Pommerenke [17,18] in the disk and first generalized to $B$ by Pfaltzgraff [14]. Pommerenke’s definition of order (the supremum of $|a_2(f)|$ over all $f$ in the family) has been generalized to higher dimensions in two ways, the (trace) order and the norm order. (See Section 2 for all related definitions.)

Graham, Kohr, and Pfaltzgraff [6] and Hamada, Honda, and Kohr [7] (see also [8]) provided generalizations of (1.1) and (1.2), respectively, to $B$ using the norm order. We provide slight alterations of their results in Section 3 before turning our attention to generalizations of both one-variable bounds using the (trace) order. While the norm order-related bounds involve the operator norm of the derivative of the mapping under consideration, the (trace) order bounds make use of its Jacobian. This is done, in part, by considering bounds on ratios between the derivative and Jacobian developed in Section 4. In Section 5, we prove a useful lower linear distortion bound for automorphisms of the ball that may be of separate interest. Our two-point distortion bounds are given in Theorems 6.1 and 6.3, and consequences of these bounds comprise the remainder of Section 6.

2 Background and notation

The norm and inner product in $C^n$ are denoted $\| \cdot \|$ and $\langle \cdot , \cdot \rangle$, respectively. The open ball centered at $a \in C^n$ of radius $r > 0$ is $B(a;r)$. (And hence $B = B(0;1)$.) The algebra of linear operators on $C^n$, endowed with operator norm, is $L(C^n)$. If $\Omega \subseteq C^n$ is open and $E \subseteq C^n$, then $H(\Omega, E)$ indicates the set of all holomorphic functions from $\Omega$ into $E$, which is given the usual topology of uniform convergence on compact subsets of $\Omega$. For $f \in H(\Omega, C^n)$, the Fréchet derivative of $f$ is $Df : \Omega \rightarrow L(C^n)$ and the Jacobian of $f$ is $J_f = \det Df$.

Let $LB(\mathbb{B}) \subseteq H(\mathbb{B}, C^n)$ be the family of all locally biholomorphic mappings. It is typical, when considering such mappings, to assume the normalization $f(0) = 0$, $Df(0) = I$, where $I = I_n \in L(C^n)$ is the identity operator, a generalization of the normalization for univalent mappings of $D$ noted in Section 1. The family of such normalized mappings is denoted $LS(\mathbb{B})$. Likewise, let $B(\mathbb{B})$ and $S(\mathbb{B})$ be the subfamilies of $LB(\mathbb{B})$ and $LS(\mathbb{B})$, respectively, consisting of biholomorphic mappings.

By $Aut \mathbb{B} \subseteq H(\mathbb{B}, \mathbb{B})$, we denote the group of biholomorphic automorphisms of $\mathbb{B}$. It is well known [20, Section 2.2] that each $\varphi \in Aut \mathbb{B}$ has the unique decomposition $\varphi = U \circ \varphi_a$, $a \in \mathbb{B}$, where $U \in L(C^n)$ is unitary and $\varphi_a \in Aut \mathbb{B}$ is the involution (i.e. $\varphi_a^{-1} = \varphi_a$) given by

$$\varphi_a(z) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a \rangle}, \quad z \in \mathbb{B}. \tag{2.1}$$

Here, $P_a$ is the orthogonal projection of $C^n$ onto $\text{span}\{a\}$ (and hence is 0 if $a = 0$), $Q_a = I - P_a$, and $s_a = \sqrt{1 - \|a\|^2}$. Note that $a = \varphi^{-1}(0)$. We also note the alternative decomposition [16] $\varphi = \varphi_b \circ V$, where $V \in L(C^n)$ is unitary and $b = \varphi(0)$.

If $f \in LB(\mathbb{B})$, then the Koebe transform of $f$ with respect to $\varphi \in Aut \mathbb{B}$ is

$$\Lambda_{\varphi}(f) = D\varphi(0)^{-1}Df(\varphi(0))^{-1}(f \circ \varphi - f(\varphi(0))).$$

Notice that $\Lambda_{\varphi}(f)$ lies in $LS(\mathbb{B})$ and lies in $S(\mathbb{B})$ if and only if $f \in B(\mathbb{B})$. A (nonempty) family $M \subseteq LS(\mathbb{B})$ is a linear-invariant family (LIF) if $\Lambda_{\varphi}(f) \in M$ for all $f \in M$ and
As mentioned in Section 1, LIFs on \( \mathbb{D} \) were introduced by Pommerenke [17,18], and generalized to \( \mathbb{B} \) by Pfaltzgraff [14]. Many of the main results regarding LIFs on the ball are collected in [5, Chapter 10] (see also [15,16]).

The LIF generated by a family \( \mathcal{F} \subseteq \mathcal{L}(\mathbb{B}) \) is

\[
\Lambda[\mathcal{F}] = \{ \Lambda_\varphi(f) : f \in \mathcal{F}, \ \varphi \in \text{Aut} \mathbb{B} \}.
\]

The argument that \( \Lambda[\mathcal{F}] \) is indeed a LIF follows from the group property

\[
\Lambda_{\varphi \psi}(f) = \Lambda_\psi(\Lambda_\varphi(f)), \quad f \in \mathcal{L}(\mathbb{B}), \ \varphi, \psi \in \text{Aut} \mathbb{B}.
\]

Note that \( f \in \mathcal{F} \) lies in \( \Lambda[\mathcal{F}] \) if and only if \( f \in \mathcal{L}(\mathbb{B}) \). In the case that \( \mathcal{F} \) consists of a single mapping \( f \), we write \( \Lambda[f] \) for the generated LIF.

As noted in Section 1, there are two notions of order of a LIF on \( \mathbb{B} \) that each reduce to Pommerenke's definition of order of a LIF on \( \mathbb{D} \) when \( n=1 \). The first is the order of a LIF \( \mathcal{M} \) introduced by Pfaltzgraff [14], given by

\[
\text{ord} \mathcal{M} = \sup \left\{ \frac{1}{2} \left| \text{tr} D^2 f(0)(u, \cdot) \right| : f \in \mathcal{M}, \ u \in \partial \mathbb{B} \right\}. \tag{2.2}
\]

This notion of order naturally leads to useful volume distortion bounds on members of a LIF (upper and lower bounds on the modulus of the Jacobian), but does not readily address linear distortion (the norm of the derivative) or growth of mappings. To remedy this, Pfaltzgraff and Suffridge [16] introduced the norm order of \( \mathcal{M} \), given by

\[
\|\text{ord}\| \mathcal{M} = \sup \left\{ \frac{1}{2} \| D^2 f(0) \| : f \in \mathcal{M} \right\}. \tag{2.3}
\]

We pause to recall that the norm of a bilinear mapping \( L: \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \) is

\[
\|L\| = \sup_{u,v \in \partial \mathbb{B}} \|L(u, v)\|.
\]

When \( L \) is symmetric (as is the case with \( D^2 f(0) \)), a result of Hörmander [10] gives

\[
\|L\| = \sup_{u \in \partial \mathbb{B}} \|L(u, u)\|.
\]

The inequality

\[
|\text{tr} A| \leq n \|A\|, \quad A \in L(\mathbb{C}^n), \tag{2.4}
\]

can then be applied to each linear operator \( D^2 f(0)(u, \cdot) \), \( u \in \partial \mathbb{B} \), to obtain the relation

\[
\text{ord} \mathcal{M} \leq n \|\text{ord}\| \mathcal{M}. \tag{2.5}
\]

The orders lie in the ranges

\[
\frac{n+1}{2} \leq \text{ord} \mathcal{M} \leq \infty, \quad 1 \leq \|\text{ord}\| \mathcal{M} \leq \infty,
\]

with all values attainable by LIFs in \( \mathcal{S}(\mathbb{B}) \). Notably, if a LIF has finite norm order, then it is a normal family [16]. To help distinguish between these definitions, we will always use \( \alpha \) to denote the (trace) order of a LIF and \( \beta \) to denote the norm order.
The infinitesimal Carathéodory metric on \( \mathbb{B} \) is \( E_{\mathbb{B}} : \mathbb{B} \times \mathbb{C}^n \to [0, \infty) \) given by
\[
E_{\mathbb{B}}(z, v) = \sup\{|Dg(z)v| : g \in H(\mathbb{B}, \mathbb{D}), \ g(z) = 0\}.
\]
We note that [5, Lemma 7.2.22] (see also [4])
\[
E_{\mathbb{B}}(z, v) = \frac{||v||^2}{1 - ||v||^2} + \frac{|\langle z, v \rangle|^2}{(1 - ||z||^2)^2}, \quad z \in \mathbb{B}, \ v \in \mathbb{C}^n.
\] (2.6)
We also consider the Carathéodory distance on \( \mathbb{B} \), a related topological metric, given by
\[
C_{\mathbb{B}}(z, w) = \sup\{d_h(g(z), g(w)) : g \in H(\mathbb{B}, \mathbb{D})\}, \quad z, w \in \mathbb{B},
\]
where \( d_h \) is the hyperbolic distance on \( \mathbb{D} \). We note the specific values
\[
C_{\mathbb{B}}(z, 0) = \tanh^{-1} ||z|| = \frac{1}{2} \log \frac{1 + ||z||}{1 - ||z||}, \quad z \in \mathbb{B},
\] (2.7)
and recall that \( C_{\mathbb{B}} \) is invariant under the action of \( \text{Aut} \mathbb{B} \). That is,
\[
C_{\mathbb{B}}(\varphi(z), \varphi(w)) = C_{\mathbb{B}}(z, w), \quad z, w \in \mathbb{B}, \ \varphi \in \text{Aut} \mathbb{B}.
\]
For more information on the Carathéodory metric and distance, see [12, Section 11.2].

3 Two-point distortion using the norm order

A generalization of (1.1) was given by Graham, Kohr, and Pfaltzgraff [6, Theorem 7] in terms of the norm order and Carathéodory distance. We give a slight modification of their result here. Note that \( C_{\mathbb{B}}, \varphi_a, \) and \( \varphi_b \) are as given in Section 2.

**Theorem 3.1.** Suppose that \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) is such that \( \beta = \|\text{ord}\| \Lambda[\mathcal{F}] < \infty \). Define the increasing function \( \psi_{n,\beta} : [0, \infty) \to [0, \infty) \) by
\[
\psi_{n,\beta}(x) = \int_0^x \frac{e^{-2(2n-1)\beta t}}{\cosh^{n-1} t} \, dt.
\]
Theorem 3.1. Suppose that \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) is such that \( \beta = \|\text{ord}\| \Lambda[\mathcal{F}] < \infty \). Define the increasing function \( \psi_{n,\beta} : [0, \infty) \to [0, \infty) \) by
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\[
\psi_{n,\beta}(x) = \int_0^x \frac{e^{-2(2n-1)\beta t}}{\cosh^{n-1} t} \, dt.
\]
Then for all \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \), we have
\[
\|f(b) - f(a)\| \geq \psi_{n,\beta}(C_{\mathbb{B}}(a, b)) \max \left\{ \frac{1}{\|Df(a)D\varphi_a(0)\|}, \frac{1}{\|Df(b)D\varphi_b(0)\|} \right\}. \quad (3.1)
\]

The related generalization of (1.2) was given by Hamada, Honda, and Kohr [7, Theorem 4.7] in the context of the unit ball of a finite-dimensional JB*-triple. We present a slight alteration of their result for \( \mathbb{B} \), although this will hold in the more general case, as well. (Note that \( \mathbb{C}^n \) is a JB*-triple.)

**Theorem 3.2.** Let \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) be such that \( \beta = \|\text{ord}\| \Lambda[\mathcal{F}] < \infty \). For all \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \), we have
\[
\|f(b) - f(a)\| \leq \frac{\exp 2\beta C_{\mathbb{B}}(a, b) - 1}{2\beta} \min\{\|Df(a)D\varphi_a(0)\|, \|Df(b)D\varphi_b(0)\|\}. \quad (3.2)
\]
Remark 3.3. The statement of [6, Theorem 7] gives (3.1) for \( f \in S(\mathbb{B}) \). Examination of the proof of that theorem reveals that an inequality established earlier in [6] is applied to \( \Lambda_{\varphi_a}(f) \), which is normalized even if \( f \) is not. Thus the normalization requirement can be removed as is given in Theorem 3.1.

Likewise, the statement of [7, Theorem 4.7] assumes \( \mathcal{F} \) is a LIF, a requirement that can be altered in a similar manner for an analogous reason.

We make two more observations regarding these two-point distortion theorems that may be illustrative when comparing them to our upcoming results.

Remark 3.4. The relative simplicity of the upper bound (3.2) in comparison to the lower bound (3.1) is due to the growth estimate [16]

\[
\|g(z)\| \leq \frac{1}{2\beta} \left( \left( \frac{1 + \|z\|}{1 - \|z\|} \right)^\beta - 1 \right), \quad z \in \mathbb{B},
\]

which follows from the upper linear distortion bound [16]

\[
\|Dg(z)\| \leq \frac{(1 + \|z\|)^{\beta-1}}{(1 - \|z\|)^{\beta+1}}, \quad z \in \mathbb{B},
\]

for \( g \) in a LIF of norm order \( \beta \). Indeed, the crux of the proof of Theorem 3.2 is to apply (3.3) to \( g = \Lambda_{\varphi_a}(f) \). There is not as appealing a lower bound known for \( \|Dg(z)w\|, w \in \mathbb{C}^n \).

A bound given in [16] is what leads to the function \( \psi_{n,\beta} \) used in Theorem 3.1.

Remark 3.5. The equalities

\[
\frac{1 - e^{-2x}}{2} = \frac{\sinh x}{\exp x}, \quad \frac{e^{2x} - 1}{2} = \exp x \sinh x, \quad x \in \mathbb{R},
\]

make clear how the two-point distortion bounds (3.1) and (3.2) reduce to the bounds (1.1) and (1.2), respectively, when \( n = 1 \). In this case, we may let \( \beta = 2 \), the maximum (norm) order of a LIF of univalent mappings of one variable. Such an observation is also noteworthy for the two-point distortion bounds given in Section 6, although the reduction will not be perfect in that scenario.

4 Bounds on derivative/Jacobian ratios

We now begin our work toward analogs of (1.1) and (1.2) using the (trace) order of a LIF. Unlike the bounds (3.1) and (3.2), these bounds will utilize the Jacobian of a mapping, which allows for different computational advantages. Furthermore, the inequality (2.5) may be used to transform these bounds to families with a given norm order.

To begin, let \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) and \( r \in (0, 1] \), and define the quantities

\[
\delta_r(\mathcal{F}) = \inf \left\{ \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} : z \in B(0; r), \ u \in \partial \mathbb{B}, \ f \in \mathcal{F} \right\}
\]

\[
\Delta_r(\mathcal{F}) = \sup \left\{ \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} : z \in B(0; r), \ u \in \partial \mathbb{B}, \ f \in \mathcal{F} \right\}.
\]
For simplicity, we write \( r(f) = \delta_r(\{f\}) \) and \( \Delta_r(f) = \Delta_r(\{f\}) \) for \( f \in \mathcal{B}(\mathbb{B}) \). Note that \( \Delta_1(\mathcal{F}) = \infty \) is allowable. In this section, we study these two quantities and provide several examples of their values for certain LIFs.

Note that, for a given nonsingular \( A \in L(\mathbb{C}^n) \),
\[
\min_{u \in \partial \mathbb{B}} \|Au\| = \sqrt{\lambda},
\]
where \( \lambda \) is the minimum eigenvalue of \( A^*A \). It follows that \( 1/\lambda \) is the maximum eigenvalue of \( (A^*A)^{-1} = A^{-1}(A^{-1})^* \). It then follows that
\[
\|A^{-1}\| = \|(A^{-1})^*\| = \frac{1}{\sqrt{\lambda}}.
\]
This allows us to rewrite
\[
\delta_r(\mathcal{F}) = \inf \left\{ \frac{1}{\|Df(z)^{-1}\| \|J_f(z)^2/(n+1)\| : z \in B(0; r), \; f \in \mathcal{F}} \right\}. \tag{4.2}
\]
On the other hand, it is clear that
\[
\Delta_r(\mathcal{F}) = \sup \left\{ \frac{\|Df(z)\|}{|J_f(z)^{2/(n+1)}| : z \in B(0; r), \; f \in \mathcal{F}} \right\}.
\]

**Example 4.1.** Let \( f: \mathbb{B} \to \mathbb{C}^n \) be the Cayley transform
\[
f(z) = \frac{z}{1-z_1},
\]
Then, in block matrix form,
\[
Df(z) = \begin{bmatrix}
\frac{1}{(1-z_1)^2} & 0 \\
\hat{z} & I_{n-1}
\end{bmatrix}, \quad z \in \mathbb{B},
\]
where \( z = (z_1, \hat{z}) \) is considered as a column vector with \( z_1 \in \mathbb{C} \) and \( \hat{z} \in \mathbb{C}^{n-1} \). Clearly, \( J_f(z) = 1/(1-z_1)^{n+1} \). It is immediate that for any \( z \in \mathbb{B} \) and \( u \in \partial \mathbb{B} \),
\[
\frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} = \|(1-z_1)^2Df(z)u\| = \|(u_1, u_1 \hat{z} + (1-z_1)\hat{u})\|.
\]
Therefore
\[
\left( \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \right)^2 = |u_1|^2 + \|u_1 \hat{z} + (1-z_1)\hat{u}\|^2 \geq |u_1|^2 + (|u_1||\hat{z}| - |1-z_1||\hat{u}|)^2. \tag{4.3}
\]
Now \( \|z\| < r \) gives

\[
\|u_1\| \|\hat{z}\| - |1 - z_1|\|\hat{u}\| \leq |u_1|\|\hat{z}\| - \left( 1 - \sqrt{r^2 - \|\hat{z}\|^2} \right) \|\hat{u}\|.
\]  \tag{4.5}

For a given \( u \in \partial \mathbb{B} \), we allow \( \|\hat{z}\| \) to vary. An elementary calculation reveals that the right-hand side of (4.5) has a maximum value of \( r - \|\hat{u}\| \), attained when \( \|\hat{z}\| = r|u_1| \). Now if \( \|\hat{u}\| > r \), then

\[
|u_1|\|\hat{z}\| - |1 - z_1|\|\hat{u}\| \leq r - \|\hat{u}\| < 0
\]

implies

\[
\left( \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \right)^2 \geq |u_1|^2 + (r - \|\hat{u}\|)^2 = 1 - 2r\|\hat{u}\| + r^2 \geq (1 - r)^2.
\]

If \( \|\hat{u}\| \leq r \), then

\[
\left( \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \right)^2 \geq |u_1|^2 \geq 1 - r^2 \geq (1 - r)^2.
\]

Since

\[
\lim_{s \to r^-} \left( \frac{\|Df(s, \hat{0})(0, \hat{u})\|}{|J_f(s, \hat{0})|^{2/(n+1)}} \right)^2 = (1 - r)^2
\]

for any \( \hat{u} \in \mathbb{C}^{n-1} \) such that \( \|\hat{u}\| = 1 \), we see that \( \delta_r(f) = 1 - r \).

Returning to (4.3) and performing an analogous calculation to that which obtained (4.4) and (4.5) yields

\[
\left( \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \right)^2 \leq |u_1|^2 + \left( |u_1|\|\hat{z}\| + (1 + \sqrt{r^2 - \|\hat{z}\|^2}) \|\hat{u}\| \right)^2.
\]

The right-hand side is again maximized when \( \|\hat{z}\| = r|u_1| \), and thus

\[
\left( \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \right)^2 \leq |u_1|^2 + (r + |\hat{u}|)^2 = 1 + 2r\|\hat{u}\| + r^2 \leq (1 + r)^2.
\]

Since

\[
\lim_{s \to r^+} \left( \frac{\|Df(s, \hat{0})(0, \hat{u})\|}{|J_f(s, \hat{0})|^{2/(n+1)}} \right)^2 = (1 + r)^2
\]

for any \( \hat{u} \in \mathbb{C}^{n-1} \) such that \( \|\hat{u}\| = 1 \), we see that \( \Delta_r(f) = 1 + r \).

**Example 4.2.** Let \( \mathcal{K}(\mathbb{B}) \) denote the family of those \( f \in \mathcal{S}(\mathbb{B}) \) such that \( f(\mathbb{B}) \) is convex, and set \( r \in (0, 1] \). To estimate \( \delta_r(\mathcal{K}(\mathbb{B})) \) and \( \Delta_r(\mathcal{K}(\mathbb{B})) \), we make use of several known distortion results for the derivatives of mappings in \( \mathcal{K}(\mathbb{B}) \) and the Jacobians of mappings in a LIF of order \( \alpha \). It is easily seen that \( \mathcal{K}(\mathbb{B}) \) is a LIF. Further, \( \|\text{ord}\| \mathcal{K}(\mathbb{B}) = 1 \). However, \( \text{ord} \mathcal{K}(\mathbb{B}) > (n + 1)/2 \) when \( n \geq 2 \), with its exact value unknown.

Gong and Liu [3] (see also [5, Theorem 7.2.26]) gave the following lower bound for \( f \in \mathcal{K}(\mathbb{B}) \) in terms of the Carathéodory metric:

\[
\|Df(z)v\| \geq \frac{1 - \|z\|}{1 + \|z\|} E_{\mathbb{B}}(z, v), \quad z \in \mathbb{B}, \ v \in \mathbb{C}^n.
\]  \tag{4.6}
In addition, Pfaltzgraff and Suffridge [16] give the following conjectured lower bound:

\[
\|Df(z)v\| \geq \frac{\|v\|}{(1 + \|z\|)^2}, \quad z \in \mathbb{B}, \ v \in \mathbb{C}^n. \tag{4.7}
\]

We also have the two-sided distortion bounds for \(f\) in a LIF of order \(\alpha\) due to Pfaltzgraff [14]:

\[
\frac{(1 - \|z\|)^{\alpha - (n+1)/2}}{(1 + \|z\|)^{\alpha + (n+1)/2}} \leq |Jf(z)| \leq \frac{(1 + \|z\|)^{\alpha - (n+1)/2}}{(1 - \|z\|)^{\alpha + (n+1)/2}}, \quad z \in \mathbb{B}. \tag{4.8}
\]

Combining the right-hand inequality with (4.6), we easily obtain, for \(f \in \mathcal{K}(\mathbb{B})\) and \(\alpha = \text{ord} \mathcal{K}(\mathbb{B})\),

\[
\frac{\|Df(z)u\|}{|Jf(z)|^{2/(n+1)}} \geq \left(1 - \|z\|\right)^{2\alpha/(n+1)+2} \geq \left(1 + \|z\|\right)^{2\alpha/(n+1)+3/2} \geq \left(1 + \|z\|\right)^{2\alpha/(n+1)+1/2}
\]

for all \(z \in \mathbb{B}\) and \(u \in \partial \mathbb{B}\), making use of (2.6). We therefore conclude

\[
\delta_r(\mathcal{K}(\mathbb{B})) \geq \frac{(1 - r)^{2\alpha/(n+1)+3/2}}{(1 + r)^{2\alpha/(n+1)+1/2}}.
\]

If the conjectured lower bound (4.7) holds, then we get the more appealing

\[
\delta_r(\mathcal{K}(\mathbb{B})) \geq \left(\frac{1 - r}{1 + r}\right)^{2\alpha/(n+1)+1}
\]

by a similar calculation.

The case of \(\Delta_r(\mathcal{K}(\mathbb{B}))\) is made easier by the estimate [5, Theorem 7.2.23]:

\[
\|Df(z)\| \leq \frac{1}{(1 - \|z\|)^2}, \quad z \in \mathbb{B}.
\]

Combining this with the left side of (4.8) gives

\[
\Delta_r(\mathcal{K}(\mathbb{B})) \leq \left(\frac{1 + r}{1 - r}\right)^{2\alpha/(n+1)+1}.
\]

While \(\text{ord} \mathcal{K}(\mathbb{B}) > (n + 1)/2\) when \(n \geq 2\), there are known linear-invariant subfamilies \(\mathcal{M} \subset \mathcal{K}(\mathbb{B})\), for which \(\text{ord} \mathcal{M} = (n + 1)/2\), the minimum possible order of a LIF. Many of these families are related to extension operators. For the simplest example, we recall the Roper–Suffridge extension operator \(\Phi\): \(\mathcal{L}S(\mathbb{D}) \rightarrow \mathcal{L}S(\mathbb{B})\) given by

\[
\Phi(f)(z) = \left( f(z_1), \sqrt{f'(z_1)} \hat{z} \right), \quad z = (z_1, \hat{z}) \in \mathbb{B},
\]

satisfies \(\Phi(\mathcal{K}(\mathbb{D})) \subset \mathcal{K}(\mathbb{B})\). (See [5, 19].) If \(\mathcal{M} = \Lambda[\Phi(\mathcal{K}(\mathbb{D}))] \subset \mathcal{K}(\mathbb{B})\), then \(\text{ord} \mathcal{M} = (n + 1)/2\). From the above calculations we see that

\[
\delta_r(\mathcal{M}) \geq \left(\frac{1 - r}{1 + r}\right)^{5/2}, \quad \Delta_r(\mathcal{M}) \leq \left(\frac{1 + r}{1 - r}\right)^2
\]

and, if the conjecture (4.7) holds (at least for \(f \in \mathcal{M}\)), then

\[
\delta_r(\mathcal{M}) \geq \left(\frac{1 - r}{1 + r}\right)^2.
\]
We conclude with a simple, yet notable, result.

**Theorem 4.3.** If $F \subseteq B(\mathbb{B})$ is compact, then $0 < \delta_r(F) \leq \Delta_r(F) < \infty$ for all $r \in (0, 1)$.

We note that $\delta_r(F) = \Delta_r(F)$ is possible when $n \geq 2$, for instance if $F$ consists solely of the identity mapping. Of course, they are both equal to 1 when $n = 1$.

**Proof of Theorem 4.3.** The result follows from the continuity of the mapping

$$(f, z, u) \mapsto \frac{\|Df(z)u\|}{|J_f(z)|^{2/(n+1)}} \in (0, \infty)$$

on the compact set $F \times \overline{B}(0; r) \times \partial \mathbb{B}$. We therefore give a brief argument demonstrating this continuity.

Let $\rho \in (0, 1 - r)$, and set $K = \overline{B}(0; r + \rho) \subseteq \mathbb{B}$. For any $f \in H(\mathbb{B}, \mathbb{C}^n)$, $z \in \overline{B}(0; r)$, and $u \in \partial \mathbb{B}$, the Cauchy integral formula gives

$$Df(z)u = \frac{1}{2\pi \rho} \int_0^{2\pi} e^{-i\theta} f(z + \rho e^{i\theta} u) \, d\theta.$$ 

Thus for $f, z, u$ as above and $g \in H(\mathbb{B}, \mathbb{C}^n)$,

$$\|Df(z)u - Dg(z)u\| \leq \frac{1}{2\pi \rho} \int_0^{2\pi} \|f(z + \rho e^{i\theta} u) - g(z + \rho e^{i\theta} u)\| \, d\theta \leq \frac{\|f - g\|_K}{\rho}.$$ 

Accordingly, $\|Df(z) - Dg(z)\| \leq \|f - g\|_K/\rho$. Further, the following inequality due to Friedland [2],

$$|\det A - \det B| \leq n\|A - B\| \max\{\|A\|^{n-1}, \|B\|^{n-1}\}, \quad A, B \in L(\mathbb{C}^n),$$

and the local uniform boundedness of $F$ give

$$|J_f(z) - J_g(z)| \leq C\|f - g\|_K, \quad f, g \in F, \quad z \in \overline{B}(0; r),$$

where $C > 0$ is a constant.

Now if $\{z_m\} \subseteq \overline{B}(0; r)$, $\{u_m\} \subseteq \partial \mathbb{B}$, and $\{f_m\} \subseteq F$ are such that $z_m \to z$, $u_m \to u$, and $f_m \to f$ uniformly on compact sets, then

$$\|Df_m(z_m)u_m - Df(z)u\| \leq \frac{\|f_m - f\|_K}{\rho} + \|Df(z_m)u_m - Df(z)u\| \to 0,$$

where the second term clearly goes to 0 using the continuity of the derivative of $f$. A similar calculation with the Jacobians gives the desired continuity. 

\[\Box\]

### 5 A lower distortion bound for automorphisms of $\mathbb{B}$

We will make use of the following lower bound in Section 6 to obtain our two-point distortion bounds. However, we believe the result may be of independent interest. Recall the notation used in (2.1).
Lemma 5.1. Let $\varphi_a \in \text{Aut}\, B$ for $a \in B$ be given by (2.1). Then for all $z \in B$ and $u \in \partial B$, we have

$$\|D \varphi_a(z)u\|^2 \geq \frac{s_a^2}{|1 - \langle z, a \rangle|^4} \times \begin{cases} s_a^2 s_z^2 & \text{if } \|z\| \leq \|a\|, \\ (1 - \|a\| \|z\|)^2 & \text{if } \|z\| > \|a\|. \end{cases} \quad (5.1)$$

If $a \neq 0$, equality is attained if and only if $z$ is a nonnegative multiple of $Q_\alpha a$ and $\|Q_\alpha a\| = \min\{\|z\|, \|a\|\}$. In particular,

$$\|D \varphi_a(z)\| \geq s_a(1 - \|a\|) \frac{1}{|1 - \langle z, a \rangle|^2}, \quad z \in B, \ u \in \partial B.$$  

Proof. Let $a, z \in B$ and $u \in \partial B$. Observe that the result is obvious if $a = 0$, for $D \varphi_0(z) = -I$. Hence we assume $a \neq 0$ for the remainder of the proof.

A direct calculation gives

$$D \varphi_a(z)u = \frac{(u, a)(a - P_\alpha z - s_a Q_\alpha z) - (1 - \langle z, a \rangle)(P_\alpha u + s_a Q_\alpha u)}{(1 - \langle z, a \rangle)^2}. \quad (5.2)$$

An application of the Pythagorean identity yields

$$|1 - \langle z, a \rangle|^4 \|D \varphi_a(z)u\|^2 = \|\langle u, a \rangle(a - P_\alpha z) - (1 - \langle z, a \rangle)P_\alpha u\|^2$$

$$+ s_a^2 \|\langle u, a \rangle Q_\alpha z + (1 - \langle z, a \rangle)Q_\alpha u\|^2.$$

We consider the first term, noting that $P_\alpha = \langle \cdot, a \rangle a/\|a\|^2$, and calculate

$$\|\langle u, a \rangle(a - P_\alpha z) - (1 - \langle z, a \rangle)P_\alpha u\|^2 = \|\langle u, a \rangle a - \frac{(u, a)a}{\|a\|^2}\|^2 = \frac{|\langle u, a \rangle|^2 s_a^4}{\|a\|^2}.$$  

Moving to the second term, we use $Q_\alpha = I - P_\alpha$ to obtain

$$\|\langle u, a \rangle Q_\alpha z + (1 - \langle z, a \rangle)Q_\alpha u\|^2$$

$$= \left\| (1 - \langle z, a \rangle)u - \frac{(u, a)a}{\|a\|^2} + \langle u, a \rangle z \right\|^2$$

$$= |1 - \langle z, a \rangle|^2 + |\langle u, a \rangle|^2 \left\| \frac{a}{\|a\|^2} - z \right\|^2$$

$$- 2 \text{Re} \left( (1 - \langle z, a \rangle)\langle u, a \rangle \left\langle u, \frac{a}{\|a\|^2} - z \right\rangle \right)$$

$$= |1 - \langle z, a \rangle|^2 + |\langle u, a \rangle|^2 \left( \frac{1}{\|a\|^2} - \frac{2 \text{Re}\langle z, a \rangle}{\|a\|^2} + \|z\|^2 \right)$$

$$+ 2 \text{Re}(1 - \langle z, a \rangle)\langle u, a \rangle \langle u, z \rangle) - \frac{2|\langle u, a \rangle|^2}{\|a\|^2} \text{Re}(1 - \langle z, a \rangle)$$

$$= |1 - \langle z, a \rangle|^2 + |\langle u, a \rangle|^2 \left( \|z\|^2 - \frac{1}{\|a\|^2} \right) + 2 \text{Re}(1 - \langle z, a \rangle)\langle a, u \rangle \langle u, z \rangle).$$
Now we may write
\[ \frac{|1 - \langle z, a \rangle|^4 \| D\varphi_a(z)u \|^2}{s^2_a} = \frac{|\langle u, a \rangle|^2 s^2_a}{\| a \|^2} + |1 - \langle z, a \rangle|^2 + \| P_a a \|^2 \left( \| z \|^2 - \frac{1}{\| a \|^2} \right) \]
\[ + 2 \Re((1 - \langle z, a \rangle)(\langle a, u \rangle, z)) \]
\[ = |1 - \langle z, a \rangle|^2 + \| P_a a \|^2 (\| z \|^2 - 1) \]
\[ + 2 \Re((1 - \langle z, a \rangle)(P_a a, z)) \]
\[ = |1 - \langle z, a \rangle + \langle z, P_a a \rangle|^2 - |\langle z, P_a a \rangle|^2 - s^2_a \| P_a a \|^2. \]

We conclude that
\[ \frac{|1 - \langle z, a \rangle|^4 \| D\varphi_a(z)u \|^2}{s^2_a} = |1 - \langle z, Q_u a \rangle|^2 - |\langle z, P_a a \rangle|^2 - s^2_a \| P_a a \|^2. \] (5.3)

We now consider (5.3) in the case that \( Q_u a = 0 \). This implies that \( a \) and \( u \) are linearly dependent, and we calculate
\[ \frac{|1 - \langle z, a \rangle|^4 \| D\varphi_a(z)u \|^2}{s^2_a} = 1 - |\langle z, Q_u a \rangle|^2 - s^2_a \| P_a a \|^2 \geq 1 - \| P_a a \|^2 = s^2_a. \]

It is not hard to see that this satisfies (5.1) with equality if and only if \( \| z \| = 0 = \| Q_u a \| \).

Now assume \( Q_u a \neq 0 \). If \( P_a a \neq 0 \), then because \( P_a a \) and \( Q_u a \) are orthogonal, we have
\[ \left\| \frac{\langle z, P_a a \rangle}{\| P_a a \|^2} \right\|^2 + \left\| \frac{\langle z, Q_u a \rangle}{\| Q_u a \|^2} \right\|^2 \leq \| z \|^2 \]
with equality if and only if \( z \in \text{span}\{P_a a, Q_u a\} \). It follows that
\[ \| z \|^2 \| P_a a \|^2 - |\langle z, P_a a \rangle|^2 \geq \frac{\| P_a a \|^2}{\| Q_u a \|^2} |\langle z, Q_u a \rangle|^2. \]

This inequality obviously also holds if \( P_a a = 0 \), and therefore we calculate
\[ \frac{|1 - \langle z, a \rangle|^4 \| D\varphi_a(z)u \|^2}{s^2_a} \]
\[ \geq |1 - \langle z, Q_u a \rangle|^2 - \| P_a a \|^2 \left( 1 - \frac{|\langle z, Q_u a \rangle|^2}{\| Q_u a \|^2} \right) \]
\[ = |1 - \langle z, Q_u a \rangle|^2 - \frac{\| a \|^2 - \| Q_u a \|^2}{\| a \|^2} (\| Q_u a \|^2 - |\langle z, Q_u a \rangle|^2) \]
\[ = s^2_a + \| Q_u a \|^2 - 2 \Re\langle z, Q_u a \rangle + \frac{\| a \|^2}{\| Q_u a \|^2} |\langle z, Q_u a \rangle|^2. \] (5.4)

To find the minimum of (5.4), we consider minimizing the real-valued function of two real variables given by
\[ f(x, y) = s^2_a + \| Q_u a \|^2 - 2x + \frac{\| a \|^2}{\| Q_u a \|^2} (x^2 + y^2), \quad x^2 + y^2 \leq \| z \|^2 \| Q_u a \|^2. \]
Clearly, the graph of $f$ as a function of two real variables is a paraboloid whose minimum value occurs at the point $(x, y) = (\|Qua\|^2/\|a\|^2, 0)$, which is an interior point of the domain provided that $\|Qua\| < \|z\||\|a\||^2$. In this case, the minimum value is

$$f\left(\frac{\|Qua\|^2}{\|a\|^2}, 0\right) = s_a^2 + \|Qua\|^2 - \frac{\|Qua\|^2}{\|a\|^2} = s_a^2 \left(1 - \frac{\|Qua\|^2}{\|a\|^2}\right).$$

(5.5)

Otherwise the minimum occurs at a point on the boundary of the domain. Clearly, this point is where $x = \|z\||\|Qua\||$ and $y = 0$, and the minimum value is

$$f(\|z\||\|Qua\||, 0) = s_a^2 + \|Qua\|^2 - 2\|z\||\|Qua\|| + \|a\|^2\|z\|^2.$$

(5.6)

Considering the right-hand side of (5.6) as a quadratic function of $\|Qua\|^2 [0, \|a\|]$, we see that

$$s_a^2 + \|Qua\|^2 - 2\|z\||\|Qua\|| + \|a\|^2\|z\|^2 \geq \begin{cases} s_a^2 s_z^2 \quad &\text{if } \|z\| \leq \|a\|, \\ (1 - \|a\||\|z\||)^2 &\text{if } \|z\| > \|a\|.
\end{cases}$$

(5.7)

Equality occurs in the case $\|z\| \leq \|a\|$ when $\|Qua\| = \|z\|$ (the vertex). In the case $\|z\| > \|a\|$, the minimum must occur when $\|Qua\| = \|a\|$, giving equality in (5.7). Under the conditions giving the interior minimum (5.5), we have that the minimum satisfies

$$s_a^2 \left(1 - \frac{\|Qua\|^2}{\|a\|^2}\right) > s_a^2 (1 - \|a\|^2\|z\|^2).$$

It is not hard to see that this interior minimum is greater than the right-hand side of (5.7) in either case, and hence that expression serves as a lower bound of $f$ in all cases. Equality is clearly attained as described in the statement of the lemma, completing the proof.

For simplicity, we note the following corollary to the proof of Lemma 5.1. Specifically, it follows immediately from (5.3).

**Corollary 5.2.** Let $\phi_a \in \text{Aut} \mathbb{B}$ for $a \in \mathbb{B}$ be given by (2.1). Then for all $z \in \mathbb{B}$, we have

$$\|D\phi_a(z)\| \leq \frac{s_a (1 + \|z||\|a\||)}{1 - \langle z, a \rangle^2}.$$  

6 Two-point distortion using the (trace) order

We will now use our work in Sections 4 and 5 to obtain lower and upper two-point distortion bounds for a family $\mathcal{F}$ of biholomorphic mappings in terms of $\text{ord} \Lambda[\mathcal{F}]$.

In preparation, we take note of the identity [20, Theorem 2.2.2]

$$\|\phi_a(z)\|^2 = 1 - \frac{s_a^2 s_z^2}{1 - \langle z, a \rangle^2}, \quad a, z \in \mathbb{B}. \quad (6.1)$$

In particular, we note for $z \in \overline{B}(0; r)$,

$$\|\phi_a(z)\|^2 \leq 1 - \frac{s_a^2 (1 - r^2)}{(1 + r||a||)^2} = \left(\frac{r + ||a||}{1 + r||a||}\right)^2. \quad (6.2)$$

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Furthermore the bound is sharp (it is attained for \( z = -ra/\|a\| \) when \( a \neq 0 \), and hence (its square root) provides the maximum value of \( \|\varphi_a(z)\| \) for \( z \in \overline{B}(0;r) \). For \( a, z \in \mathbb{B} \), we define the notation
\[
 r(a, z) = \frac{\|\varphi_a(z)\| + \|a\|}{1 + \|a\|\|\varphi_a(z)\|} \tag{6.3}
\]
We recall [20] that \( \varphi_a(\overline{B}(0;\varepsilon)) \), \( 0 < \varepsilon < 1 \), is a (closed) ellipsoid, and thus \( r(a, z) \) is the maximum norm over the ellipsoid corresponding to \( \varepsilon = \|\varphi_a(z)\| \). Since \( \varphi_a \) is an involution, this means \( r(a, z) \) is the maximum norm over the ellipsoid \( \varphi_a(\overline{B}(0;\varepsilon)) \) where \( \varepsilon \) is such that \( z \in \partial \varphi_a(\overline{B}(0;\varepsilon)) \).

**Theorem 6.1.** Let \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) be such that \( \alpha = \text{ord} \Lambda[\mathcal{F}] < \infty \). For all \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \), we have
\[
 \|f(b) - f(a)\| \geq \frac{n+1}{4\alpha} \left( 1 - \exp \left( -4\alpha C_{\mathcal{B}}(a, b) \right) \right) \nonumber \\
 \times \max \left\{ \delta_{r(a,b)}(\mathcal{F}) s_a(1 - \|a\|) |J_f(a)|^{2/(n+1)} , \right. \nonumber \\
\left. \delta_{r(b,a)}(\mathcal{F}) s_b(1 - \|b\|) |J_f(b)|^{2/(n+1)} \right\} \tag{6.4}
\]

**Proof.** Let \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \). Write \( g = \Lambda \varphi_a(f) \in \Lambda[\mathcal{F}] \). We apply Lemma 5.1 and (4.1) to calculate for \( z \in \mathbb{B} \) and \( u \in \partial \mathbb{B} \),
\[
 \|Df(a)D\varphi_a(0)Dg(z)u\| = \|Df(\varphi_a(z))D\varphi_a(z)u\| 
\geq \frac{\|D\varphi_a(z)u\|}{\|Df(\varphi_a(z))^{-1}\|} 
\geq \frac{\|D\varphi_a(z)\|}{\|Df(\varphi_a(z))^{-1}\|} \left( 1 - \|z,a\|^2 \right). 
\]

Now we note that \( D(\varphi_a \circ \varphi_z)(0) = D\varphi_a(z)D\varphi_z(0) \). We may write \( \varphi_a \circ \varphi_z = \varphi_p \circ U \), where \( U \) is a unitary operator on \( \mathbb{C}^n \) and \( p = \varphi_a(z) \). It follows that
\[
 |J_{\varphi_a \circ \varphi_z}(0)| = |J_{\varphi_p}(0)| = (1 - \|\varphi_a(z)\|^{2(n+1)})/2. 
\]
(We use that \( J_{\varphi_p}(0) = s_p^{n+1} \) for \( p \in \mathbb{B} \).) Therefore
\[
 |J_{\varphi_a}(z)| = \left| \frac{J_{\varphi_a \circ \varphi_z}(0)}{J_{\varphi_z}(0)} \right| = \left( 1 - \|\varphi_a(z)\|^{2(n+1)} \right)/(s_z^2(1 - \langle z,a\rangle)^2), 
\]
where the last equality follows from (6.1). This allows for the calculation
\[
 |J_f(z)| = \frac{|J_f(\varphi_a(z))| |J_{\varphi_a}(z)|}{|J_{\varphi_z}(0)| |J_f(a)|} = \frac{|J_f(\varphi_a(z))|}{|J_f(a)|} \left( 1 - \langle z,a\rangle \right)^{n+1}. \tag{6.5}
\]
Assume that \( \|z\| \leq \|\varphi_a(b)\| \). From (6.2) and (6.3) we see that \( \|\varphi_a(z)\| \leq r(a, b) \). Putting the above together, we have
\[
 \frac{\|Df(a)D\varphi_a(0)Dg(z)u\|}{|J_f(z)|^{2/(n+1)}} \geq \frac{s_a(1 - \|a\|) |J_f(a)|^{2/(n+1)}}{\|Df(\varphi_a(z))^{-1}\| |J_f(\varphi_a(z))|^{2/(n+1)}} 
\geq \delta_{r(a,b)}(\mathcal{F}) s_a(1 - \|a\|) |J_f(a)|^{2/(n+1)}. \tag{6.6}
\]
Now use the lower bound in (4.8) to obtain

\[
\|Df(a)D\varphi_a(0)Dg(z)u\| \geq \delta_{r(a,b)}(F)s_a(1 - \|a\|)|J_f(a)|^{2/(n+1)}|J_g(z)|^{2/(n+1)} \\
\geq \delta_{r(a,b)}(F)s_a(1 - \|a\|)|J_f(a)|^{2/(n+1)}(1 - \|z\|)^{2\alpha/(n+1)-1} \\
(1 + \|z\|)^{2\alpha/(n+1)+1}
\]

(6.7)

for all \(z \in \mathbb{B}\) such that \(\|z\| \leq \|\varphi_a(b)\|\) and all \(u \in \partial \mathbb{B}\).

Fix \(z \in \mathbb{B}\) such that \(\|z\| \leq \|\varphi_a(b)\|\), and let \(z_0 \in \partial B(0; \|z\|)\) such that

\[
\rho = \|Df(a)D\varphi_a(0)g(z_0)\| = \min\{\|Df(a)D\varphi_a(0)g(w)\| : w \in \partial B(0; \|z\|)\} > 0.
\]

It follows that

\[
\mathcal{B}(0; \rho) \subseteq Df(a)D\varphi_a(0)g(\mathcal{B}(0; \|z\|)).
\]

Now let \(\Gamma: [0, 1] \to \mathcal{B}(0; \rho)\) be given by

\[
\Gamma(t) = tDf(a)D\varphi_a(0)g(z_0),
\]

and let \(\gamma: [0, 1] \to \mathcal{B}(0; \|z\|)\) be given by

\[
\gamma(t) = g^{-1}(tg(z_0)) = g^{-1}(D\varphi_a(0)^{-1}Df(a)^{-1}\Gamma(t)).
\]

Note that \(\gamma\) parameterizes a contour from 0 to \(z_0\). We then have, noting that (6.7) holds with \(\gamma(t), 0 \leq t \leq 1\), in place of \(z\),

\[
\|f(\varphi_a(z)) - f(a)\| = \|Df(a)D\varphi_a(0)g(z)\| \\
\geq \|Df(a)D\varphi_a(0)g(z_0)\| \\
= \int_{\Gamma} \|dw\| \\
= \int_{\gamma} \|Df(a)D\varphi_a(0)Dg(w)dw\| \\
= \int_{\gamma} \|Df(a)D\varphi_a(0)Dg(w)\| \frac{dw}{\|dw\|} \|dw\| \\
\geq \delta_{r(a,b)}(F)s_a(1 - \|a\|)|J_f(a)|^{2/(n+1)} \int_{\gamma} \frac{(1 - \|w\|)^{2\alpha/(n+1)-1}}{(1 + \|w\|)^{2\alpha/(n+1)+1}} \|dw\|.
\]

We note that the differential inequality \(\|dw\| \geq d\|w\|\) follows from the triangle inequality. The last integral in the above can be evaluated as

\[
\int_{0}^{\|z\|} \left(\frac{1 - t}{1 + t}\right)^{2\alpha/(n+1)-1} \frac{dt}{(1 + t)^2} = \frac{n + 1}{4\alpha} \left(1 - \left(\frac{1 - \|z\|}{1 + \|z\|}\right)^{2\alpha/(n+1)}\right).
\]

Use (2.7) to obtain

\[
\|f(\varphi_a(z)) - f(a)\| \geq \frac{n + 1}{4\alpha} \delta_{r(a,b)}(F)s_a(1 - \|a\|)|J_f(a)|^{2/(n+1)} \left(1 - \exp \frac{-4\alpha C B(0, z)}{n + 1}\right).
\]

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Now substitute \( z = \varphi_a(b) = \varphi_a^{-1}(b) \). It follows that \( C_{\mathcal{B}}(a, b) = C_{\mathcal{B}}(0, z) \) because \( C_{\mathcal{B}} \) is invariant under the action of \( \text{Aut} \mathcal{B} \). This gives

\[
\|f(b) - f(a)\| \geq \frac{n + 1}{4 \alpha} \delta_{r(a,b)}(\mathcal{F}) s_a(1 - \|a\|) |J_f(a)|^{2/(n+1)} \left( 1 - \exp \frac{-4 \alpha C_{\mathcal{B}}(a, b)}{n + 1} \right).
\]

The same inequality with \( b \) and \( a \) interchanged clearly holds, and combining these results completes the proof. \( \square \)

**Remark 6.2.** Using an argument relying on isolation of zeros, Kim and Minda [11] showed that any nonconstant \( f \in H(B, C) \) satisfying (1.1) must be univalent. Such a result is easily seen not to hold when \( n \geq 2 \) for the bound in Theorem 6.1. Indeed, if \( f \in H(B, C^n) \) is a nonzero singular linear operator, then \( f \) is not biholomorphic but the bound still holds. As no zeros of members of \( H(B, C) \) are isolated when \( n \geq 2 \), a proof like that of Kim and Minda is not available to us, for if the zeros of \( J_f \) are assumed isolated, then \( J_f(z) \neq 0 \) for all \( z \in B \), \( f \) is locally biholomorphic, and it is immediate that satisfaction of the bound in Theorem 6.1 guarantees \( f \) is biholomorphic.

We now consider the upper two-point distortion bound that serves as a partner of the lower bound in Theorem 6.1.

**Theorem 6.3.** Let \( \mathcal{F} \subseteq \mathcal{B}(B) \) be such that \( \alpha = \text{ord}\Lambda[\mathcal{F}] < \infty \). For all \( f \in \mathcal{F} \) and \( a, b \in B \), we have

\[
\|Df(a)D\varphi_a(0)Dg(z)\| \leq \frac{n + 1}{4 \alpha} \left( \exp \frac{4 \alpha C_{\mathcal{B}}(a, b)}{n + 1} - 1 \right) \times \min \left\{ \Delta_{r(a,b)}(\mathcal{F}) s_a(1 + \|a\|) |J_f(a)|^{2/(n+1)}, \Delta_{r(b,a)}(\mathcal{F}) s_b(1 + \|b\|) |J_f(b)|^{2/(n+1)} \right\}.
\]

**Proof.** Let \( f, a, \) and \( g \) be as in the proof of Theorem 6.1. By Corollary 5.2, we have

\[
\|Df(a)D\varphi_a(0)Dg(z)\| = \|Df(\varphi_a(z))D\varphi_a(z)\| \leq \frac{s_a(1 + \|a\|) \|Df(\varphi_a(z))\|}{|1 - \langle z, a \rangle|^2}
\]

for \( z \in B \). We assume that \( \|z\| \leq \|\varphi_a(b)\| \) so that \( \|\varphi_a(z)\| \leq r(a, b) \). Using (6.5), we have

\[
\frac{\|Df(a)D\varphi_a(0)Dg(z)\|}{|J_g(z)|^{2/(n+1)}} \leq \frac{s_a(1 + \|a\|) \|Df(\varphi_a(z))\| |J_f(a)|^{2/(n+1)}}{\|J_f(\varphi_a(z))\|^{2/(n+1)}} \leq \Delta_{r(a,b)}(\mathcal{F}) s_a(1 + \|a\|) |J_f(a)|^{2/(n+1)}.
\]

We use the upper bound in (4.8) to calculate

\[
\|Df(a)D\varphi_a(0)Dg(z)\| \leq \Delta_{r(a,b)}(\mathcal{F}) s_a(1 + \|a\|) |J_f(a)|^{2/(n+1)} |J_g(z)|^{2/(n+1)} \leq \Delta_{r(a,b)}(\mathcal{F}) s_a(1 + \|a\|) |J_f(a)|^{2/(n+1)} \left( 1 + \|z\| \right)^{2\alpha/(n+1)-1} \left( 1 - \|z\| \right)^{2\alpha/(n+1)+1}.
\]
for all \( z \in \mathbb{B} \) such that \( \|z\| \leq \|\varphi_a(b)\| \).

Now fix \( z \in \mathbb{B} \) with \( \|z\| \leq \|\varphi_a(b)\| \), and define \( G: [0, 1] \to \mathbb{R} \) by
\[
G(t) = \text{Re} \left( Df(a)D\varphi_a(0)g(tz), \frac{Df(a)D\varphi_a(0)g(tz)}{\|Df(a)D\varphi_a(0)g(tz)\|} \right).
\]

For \( t \in [0, 1] \), we calculate
\[
G'(t) = \text{Re} \left( Df(a)D\varphi_a(0)Dg(tz)z, \frac{Df(a)D\varphi_a(0)g(tz)}{\|Df(a)D\varphi_a(0)g(tz)\|} \right)
\]
\[
\leq \|Df(a)D\varphi_a(0)Dg(tz)z\| \|z\|.
\]

Noting that \( G(0) = 0 \), we find
\[
\|f(\varphi_a(z)) - f(a)\|
= \|Df(a)D\varphi_a(0)g(z)\|
= G(1)
= \int_0^1 G'(t) \, dt
\leq \Delta_{r(a,b)}(F)s_a(1 + \|a\|)|J_f(a)|^{2/(n+1)} \left( \frac{1 + t\|z\|}{1 - t\|z\|} \right)^{2\alpha/(n+1) - 1} \frac{\|z\| \, dt}{(1 - t\|z\|)^2}
\]
\[
= \frac{n + 1}{4\alpha} \Delta_{r(a,b)}(F)s_a(1 + \|a\|)|J_f(a)|^{2/(n+1)} \left( \frac{1 + \|z\|}{1 - \|z\|} \right)^{2\alpha/(n+1) - 1}.
\]

We now make use of (2.7) to write
\[
\|f(\varphi_a(z)) - f(a)\|
\leq \frac{n + 1}{4\alpha} \Delta_{r(a,b)}(F)s_a(1 + \|a\|)|J_f(a)|^{2/(n+1)} \left( \exp \frac{4\alpha C_{\mathbb{B}}(0, z)}{n + 1} - 1 \right).
\]

Substitute \( z = \varphi_a(b) (= \varphi_a^{-1}(b)) \) and use the invariance of \( C_{\mathbb{B}} \) under the action of \( \text{Aut} \mathbb{B} \) to see that
\[
\|f(b) - f(a)\|
\leq \frac{n + 1}{4\alpha} \Delta_{r(a,b)}(F)s_a(1 + \|a\|)|J_f(a)|^{2/(n+1)} \left( \exp \frac{4\alpha C_{\mathbb{B}}(a, b)}{n + 1} - 1 \right).
\]

The same inequality with \( b \) and \( a \) interchanged clearly holds, and combining these results completes the proof. \( \square \)

**Remark 6.4.** A number of estimates are used in the proofs of Theorems 6.1 and 6.3. These include upper and lower bounds on \( \|D\varphi_a(z)u\| \) provided by Lemma 5.1 and Corollary 5.2 as well as the inequalities
\[
\delta_{r(a,b)}(F) \leq \frac{1}{\|Df(\varphi_a(z)) - 1\| |J_f(\varphi_a(z))|^{2/(n+1)}}, \quad \Delta_{r(a,b)}(F) \geq \frac{\|Df(\varphi_a(z))\|}{|J_f(\varphi_a(z))|^{2/(n+1)}}
\]
in lines (6.6) and (6.9). It is certainly the case that these estimates destroy any sharpness to the two-point distortion bounds for any nontrivial mappings. However, replacing them
with something tighter, for instance using \( \delta_{\|\varphi_a(\cdot)\|}^\text{ord}(\mathcal{F}) \) in place of \( \delta_{r(a,b)}(\mathcal{F}) \) in (6.6), would add significant complication and prevent the clean evaluations of the integrals computed in the proofs of Theorems 6.1 and 6.3.

We certainly leave open the possibility that superior two-point distortion bounds could be obtained by considering a different approach.

The lower and upper bounds on the Jacobian of a member of a LIF of norm order \( \beta \ [5, \text{Corollary 10.4.6}] \) follow from (2.5). Repeating the proofs of Theorems 6.1 and 6.3, using those bounds at (6.7) and (6.10), results in the following.

**Corollary 6.5.** Let \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) be such that \( \beta = \|\text{ord}\| \Lambda[\mathcal{F}] < \infty \). For all \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \), we have

\[
\|f(b) - f(a)\| \geq \frac{n + 1}{4n\beta} \left(1 - \exp \frac{-4n\beta C_\mathbb{B}(a, b)}{n + 1}\right) \times \max \left\{ \delta_{r(a,b)}(\mathcal{F}) s_a (1 - \|a\|) |J_f(a)|^{2/(n+1)}, \right. \\
\left. \delta_{r(b,a)}(\mathcal{F}) s_b (1 - \|b\|) |J_f(b)|^{2/(n+1)} \right\},
\]

and

\[
\|f(b) - f(a)\| \leq \frac{n + 1}{4n\beta} \left(\exp \frac{4n\beta C_\mathbb{B}(a, b)}{n + 1} - 1\right) \times \min \left\{ \Delta_{r(a,b)}(\mathcal{F}) s_a (1 + \|a\|) |J_f(a)|^{2/(n+1)}, \right. \\
\left. \Delta_{r(b,a)}(\mathcal{F}) s_b (1 + \|b\|) |J_f(b)|^{2/(n+1)} \right\}.
\]

**Remark 6.6.** It is interesting to compare the two-point distortion bounds obtained in terms of norm order in Corollary 6.5 with those of Section 3. Since there is no upper bound on the norm of an operator in terms of its determinant (a type of reversal of the well-known \( |\det A| \leq \|A\|^n \) for \( A \in L(\mathbb{C}^n) \)), the bounds (3.1) and (3.2) cannot immediately be used to give bounds in terms of the Jacobian, as we have in the above. However, we turn to the ratios \( \delta_r(f) \) and \( \Delta_r(f) \) for help. Noting that \( D\varphi_a(0) = -s_a^2 P_a - s_a Q_a \) for \( a \in \mathbb{B} \) (see [20, Theorem 2.2.2] or (5.2)), we have \( \|D\varphi_a(0)\| = s_a \) and \( \|D\varphi_a(0)^{-1}\| = 1/s_a^2 \).

Application of this to (3.1) and (3.2) and elementary calculations result in

\[
\|f(b) - f(a)\| \geq \psi_{n,\beta}(C_\mathbb{B}(a, b)) \max \left\{ \delta_{\|a\|}(f) s_a^2 |J_f(a)|^{2/(n+1)}, \delta_{\|b\|}(f) s_b^2 |J_f(b)|^{2/(n+1)} \right\}
\]

and

\[
\|f(b) - f(a)\| \leq \frac{\exp 2\beta C_\mathbb{B}(a, b) - 1}{2\beta} \min \left\{ \Delta_{\|a\|}(f) s_a |J_f(a)|^{2/(n+1)}, \Delta_{\|b\|}(f) s_b |J_f(b)|^{2/(n+1)} \right\}
\]

for \( f \in \mathcal{B}(\mathbb{B}) \) with \( \beta = \|\text{ord}\| \Lambda[f] < \infty \) and \( a, b \in \mathbb{B} \). It is not surprising that the latter is superior to (6.12), as it avoids several estimates, including \( \alpha \leq n/\beta \). Notably, (6.11) avoids use of the function \( \psi_{n,\beta} \), which is advantageous.

Letting \( a = z \) and \( b = 0 \) in (6.4) and (6.8), using (2.7) and that \( r(0, z) = \|z\| \), we obtain the following growth bounds for biholomorphic mappings.
Corollary 6.7. Let \( f \in \mathcal{B}(\mathbb{B}) \) be such that \( f(0) = 0 \) and \( \alpha = \text{ord} \Lambda[f] < \infty \). Then for all \( z \in \mathbb{B} \), we have

\[
\|f(z)\| \geq \frac{n + 1}{4\alpha} \delta_{\|z\|}(f) J_f(0)^{-2/(n+1)} \left( 1 - \frac{1 - \|z\|}{1 + \|z\|} \right)^{2\alpha/(n+1)} \tag{6.13}
\]

and

\[
\|f(z)\| \leq \frac{n + 1}{4\alpha} \Delta_{\|z\|}(f) J_f(0)^{-2/(n+1)} \left( \frac{1 + \|z\|}{1 - \|z\|} \right)^{2\alpha/(n+1)} - 1 \tag{6.14}
\]

In the commonly considered case that \( f \in \mathcal{S}(\mathbb{B}) \), the above is simplified since \( J_f(0) = 1 \). These growth bounds lead to alternative two-point distortion results in terms of derivative norms.

Theorem 6.8. Let \( \mathcal{F} \subseteq \mathcal{B}(\mathbb{B}) \) be such that \( \alpha = \text{ord} \Lambda[\mathcal{F}] < \infty \). For all \( f \in \mathcal{F} \) and \( a, b \in \mathbb{B} \), we have

\[
\|f(b) - f(a)\| \geq \frac{n + 1}{4\alpha} \delta_{\|\varphi_a(b)\|}(\Lambda[\mathcal{F}]) \left( 1 - \exp \frac{-4\alpha C_\mathbb{B}(a, b)}{n + 1} \right) \times \max \left\{ \frac{1}{\| (Df(a)D\varphi_a(0))^{-1} \|}, \frac{1}{\| (Df(b)D\varphi_b(0))^{-1} \|} \right\} \tag{6.15}
\]

and

\[
\|f(b) - f(a)\| \leq \frac{n + 1}{4\alpha} \Delta_{\|\varphi_a(b)\|}(\Lambda[\mathcal{F}]) \left( \exp \frac{4\alpha C_\mathbb{B}(a, b)}{n + 1} - 1 \right) \times \min \{\| Df(a)D\varphi_a(0) \|, \| Df(b)D\varphi_b(0) \| \}. \tag{6.16}
\]

Proof. Let \( f \in \mathcal{F} \) and \( g = \Lambda\varphi_a(f) \in \Lambda[\mathcal{F}] \). Apply (6.13) to \( g \), and use (2.7) and the inequality

\[
\|g(z)\| \leq \| (Df(a)D\varphi_a(0))^{-1} \| f(\varphi_a(z)) - f(a) \|, \quad z \in \mathbb{B},
\]

to see that for all \( z \in \mathbb{B} \),

\[
\|f(\varphi_a(z)) - f(a)\| \geq \frac{n + 1}{4\alpha} \delta_{\|z\|}(g) \left( 1 - \exp \frac{-4\alpha C_\mathbb{B}(0, z)}{n + 1} \right) \frac{1}{\| (Df(a)D\varphi_a(0))^{-1} \|}.
\]

Let \( z = \varphi_a(b) = \varphi_a^{-1}(b) \) and use the invariance of \( C_\mathbb{B} \) under the action of Aut \( \mathbb{B} \) to obtain

\[
\|f(b) - f(a)\| \geq \frac{n + 1}{4\alpha} \delta_{\|\varphi_a(b)\|}(\Lambda[\mathcal{F}]) \left( 1 - \exp \frac{-4\alpha C_\mathbb{B}(a, b)}{n + 1} \right) \frac{1}{\| (Df(a)D\varphi_a(0))^{-1} \|}.
\]

The same inequality with \( a \) and \( b \) interchanged clearly holds, and since \( \|\varphi_a(b)\| = \|\varphi_b(a)\| \) (see (6.1)), we can conclude the lower bound as given in the theorem.

The proof of the upper bound travels a similar path, noting that

\[
\|g(z)\| \geq \frac{\|f(\varphi_a(z)) - f(a)\|}{\| Df(a)D\varphi_a(0) \|}, \quad z \in \mathbb{B},
\]

follows from (4.1). \qed
Remark 6.9. The respective similarity between the bounds (6.15) and (6.16) and the bounds (3.1) and (3.2) is apparent. Since $\text{tr } A$ can equal 0 for nonzero $A \in L(C^n)$, there is no inequality similar to (2.4) bounding $\|A\|$ above by a positive multiple of $|\text{tr } A|$. Accordingly, there is no easily found reversal of (2.5). Therefore a given value of $\alpha = \text{ord } A[\mathcal{F}]$ does not supply a bound on $\beta = \|\text{ord } A[\mathcal{F}]$ allowing use of (3.1) or (3.2) to obtain bounds of the type given in Theorem 6.8 in the same manner as the bounds of Theorems 6.1 and 6.3 were easily applied to obtain those of Corollary 6.5.

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References


