A Class of Loewner Chain Preserving Extension Operators

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Abstract

We consider operators that extend locally univalent mappings of the unit disk \( \Delta \) in \( \mathbb{C} \) to locally biholomorphic mappings of the Euclidean unit ball \( B \) of \( \mathbb{C}^n \). For such an operator, we seek conditions under which \( e^t \Phi(e^{-t}f(\cdot,t)), t \geq 0 \), is a Loewner chain on \( B \) whenever \( f(\cdot,t), t \geq 0 \), is a Loewner chain on \( \Delta \). We primarily study operators of the form 

\[
G; f(z) = (f(z_1) + G([f'(z_1)]^\beta \bar{z}), [f'(z_1)]^\beta \bar{z}), \bar{z} = (z_2, \ldots, z_n),
\]

where \( \beta \in [0,1/2] \) and \( G: \mathbb{C}^{n-1} \to \mathbb{C} \) is holomorphic, finding that, for \( \Phi;G,\beta \) to preserve Loewner chains, the maximum degree of terms appearing in the expansion of \( G \) is a function of \( \beta \). Further applications involving Bloch mappings and radius of starlikeness are given, as are elementary results concerning extreme points and support points.

1 Preliminaries and notation

We continue the study of extension operators that began with the work of Roper and Suffridge [16]. Since the introduction of what is now called the Roper–Suffridge extension operator, several modifications of that operator have been examined (for instance, in [3,6,9]) to determine when the extension of a one variable mapping with a particular geometric property has the analogous property in several variables. A good deal of this analysis involves the use of Loewner chains, and that motivates this work.

Our setting is \( \mathbb{C}^n \), the space of \( n \in \mathbb{N} \) complex variables, equipped with the Euclidean inner product \( \langle z, w \rangle = \sum_{k=1}^{n} z_k \bar{w}_k \), associated norm \( \|z\| = \langle z, z \rangle^{1/2} \) for \( z, w \in \mathbb{C}^n \), and canonical basis \( \{e_1, \ldots, e_n\} \). We let \( B_n \) denote the open unit ball of \( \mathbb{C}^n \), writing \( B \) when the dimension is understood, and let \( \Delta \) be the open unit disk of \( \mathbb{C} \). It is often convenient to write a vector \( z \in \mathbb{C}^n \) as \( z = (z_1, \tilde{z}) \), where \( z_1 \in \mathbb{C} \) and \( \tilde{z} \in \mathbb{C}^{n-1} \). If \( E \subseteq \mathbb{C}^n \) and \( r \geq 0 \), we write \( rE = \{rz : z \in E\} \).

For an open set \( \Omega \subseteq \mathbb{C}^n \), let \( H(\Omega, \mathbb{C}^m) \), \( m \in \mathbb{N} \), denote the space of all holomorphic mappings from \( \Omega \) into \( \mathbb{C}^m \), endowed with the compact-open topology. This is the topology of local uniform convergence (uniform convergence on compact sets) on \( \Omega \) and makes \( H(\Omega, \mathbb{C}^m) \) a locally convex topological vector space. Let \( \mathcal{L}S_n \) denote the family of all \( F \in H(B_n, \mathbb{C}^n) \) that are locally biholomorphic and normalized so that \( F(0) = 0 \) and \( DF(0) = I \), where \( DF \) is the Frechét derivative of \( F \) and \( I \) is the identity operator on \( \mathbb{C}^n \). The family of those

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$F \in \mathcal{L}S_n$ that are biholomorphic on $B$ is written $S_n$. It follows that $S_1$ is the classical family of schlicht mappings of $\Delta$. We will also consider the geometric families

$$S^*_n = \{ F \in S_n : F(B) \text{ is starlike with respect to } 0 \}$$

$$K_n = \{ F \in S_n : F(B) \text{ is convex} \}.$$  

Let $\Omega \subseteq \mathbb{C}^n$ be open with $0 \in \Omega$. It will be useful to consider two types of expansions of a function $F \in H(\Omega, \mathbb{C}^m)$, $m \in \mathbb{N}$, about 0. If $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $\mathbb{N}_0^n$ is the set of multi-indices. We adopt the traditional notation for $z \in \mathbb{C}^n$: $|\alpha| = \sum_{k=1}^n \alpha_k$ and $z^\alpha = \prod_{k=1}^n z_k^{\alpha_k}$. We may then write

$$F(z) = \sum_{\alpha \in \mathbb{N}_0^n} z^{\alpha} a_{\alpha},$$

where $a_\alpha \in \mathbb{C}^m$ for each $\alpha \in \mathbb{N}_0^n$. The series converges absolutely and locally uniformly in a neighborhood of 0 in $\Omega$. (When $\Omega$ is either $B$ or $\mathbb{C}^n$, this neighborhood is all of $\Omega$.) For $j \in \mathbb{N}_0$, define $P_j \in H(\mathbb{C}^n, \mathbb{C}^m)$ by $P_j(z) = \sum_{\alpha \in \mathbb{N}_0^n : |\alpha| = j} z^{\alpha} a_{\alpha}$. Then $P_j$ is a homogeneous polynomial of degree $j$ (meaning $P_j(\lambda z) = \lambda^j P_j(z)$ for $z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$) and we have the homogeneous expansion

$$F(z) = \sum_{j=0}^{\infty} P_j(z),$$

valid for $z$ in a neighborhood of 0. We denote the space of all homogeneous polynomials of degree $j$ from $\mathbb{C}^n$ into $\mathbb{C}$ by $\mathcal{P}_j(n)$. With the norm

$$\|P\| = \sup_{u \in \partial B} |P(u)|, \quad P \in \mathcal{P}_j(n),$$

$\mathcal{P}_j(n)$ is a Banach space. In addition, the bound $|P(z)| \leq \|P\| \|z\|^j$ holds for all $P \in \mathcal{P}_j(n)$ and $z \in \mathbb{C}^n$.

When considering functions $F : B \times [0, \infty) \to \mathbb{C}^n$, we adopt a few typical notational conventions. We write $DF(z, t)$ to mean the “partial” Fréchet derivative of $F$ at $z$ with $t$ fixed. (In other words, if, for fixed $t \geq 0$, $F_t = F(\cdot, t)$, then $DF(z, t) = DF_t(z)$. This is denoted $F'(z, t)$ when $n = 1$.) It will also be convenient to write $F(\cdot, 0)$ when the context is clear, rather than to name a new function.

A Loewner chain is a function $F : B \times [0, \infty) \to \mathbb{C}^n$ such that for all $t \geq 0$, $F(\cdot, t)$ is biholomorphic, $F(0, t) = 0$, $DF(0, t) = e^t I$, and $F(\cdot, s) \prec F(\cdot, t)$ for all $t \geq s \geq 0$. Here the symbol $\prec$ refers to subordination. It is therefore the case that for all $t \geq s \geq 0$, there is a holomorphic mapping $v_{s,t} : B \to B$ with $v_{s,t}(0) = 0$ (a so-called “Schwarz mapping”) such that $F(z, s) = F(v_{s,t}(z), t)$ for all $z \in B$. In addition, a Loewner chain $F$ is a locally Lipschitz continuous (and hence locally absolutely continuous) function of $t$ locally uniformly with respect to $z$.

A mapping $F \in S_n$ is said to have parametric representation if there is a Loewner chain $F : B \times [0, \infty) \to \mathbb{C}^n$ such that $F = F(\cdot, 0)$ and if the family $\{ e^{-t} F(\cdot, t) : t \geq 0 \}$ is normal. Denote the family of all mappings $F \in S_n$ with parametric representation by $S^0_n$. It is well known that $S^0_1 = S_1$, but $S^0_n \subsetneq S_n$ if $n \geq 2$. The latter result is evident as $S^0_n$ is a.
compact family, but $S_n$ is not. A discussion of these topics is provided in Chapter 8 of the monograph [4] of Graham and Kohr.

The following criterion is useful for constructing Loewner chains. It is a modification of a result of Pfaltzgraff [14] given in [3]. We remark that “measurable” means with respect to Lebesgue measure on $\mathbb{R}$.

**Theorem 1.1.** Let $F : B \times [0, \infty) \to \mathbb{C}^n$ be such that

(a) $F(0,t) = 0$ and $DF(0,t) = e^t I$ for all $t \geq 0$,

(b) $F(\cdot,t)$ is holomorphic for all $t \geq 0$, and

(c) $F(z,t)$ is a locally absolutely continuous function of $t \in [0,\infty)$ locally uniformly with respect to $z \in B$.

Let $h : B \times [0, \infty) \to \mathbb{C}^n$ be such that

(a) $h(0,t) = 0$ and $Dh(0,t) = I$ for all $t \geq 0$,

(b) $h(\cdot,t)$ is holomorphic for all $t \geq 0$,

(c) $h(z,\cdot)$ is measurable for each $z \in B$, and

(d) $\text{Re}(h(z,t),z) > 0$ for all $z \in B \setminus \{0\}$ and $t \geq 0$.

If, for all $z \in B$ and almost every $t \in [0,\infty)$,

$$\frac{\partial F}{\partial t}(z,t) = DF(z,t)h(z,t)$$

and if, for some increasing sequence $\{t_m\}_{m=1}^\infty \subseteq [0,\infty)$ with $t_m \to \infty$, there is some $G \in H(B,\mathbb{C}^n)$ such that

$$\lim_{m \to \infty} e^{-t_m} F(\cdot,t_m) = G$$

locally uniformly on $B$, then $F$ is a Loewner chain.

## 2 Extension operators

For the remainder of this article, unless otherwise noted, assume that $n \geq 2$. We say that a function $\Phi : \mathcal{L}S_1 \to \mathcal{L}S_n$ is an extension operator if $\Phi$ is continuous (with respect to the compact-open topologies of $\mathcal{L}S_1$ and $\mathcal{L}S_n$) and if, for each $f \in \mathcal{L}S_1$,

$$[\Phi(f)](\zeta e_1) = f(\zeta)e_1, \quad \zeta \in \Delta.$$

We call an extension operator $\Phi$ Loewner chain preserving provided that, whenever $f : \Delta \times [0,\infty) \to \mathbb{C}$ is a Loewner chain, the function $F : B \times [0,\infty) \to \mathbb{C}^n$ given by

$$F(\cdot,t) = e^t \Phi(e^{-t} f(\cdot,t)), \quad t \geq 0,$$

is also a Loewner chain.

A benefit provided by the application of Loewner chains to the analysis of extension operators is a consequence of the following theorem of Pfaltzgraff and Suffridge [15]:

$$\text{...}$$
Theorem 2.1. Let $F \in \mathcal{L}S_n$. Then $F \in S^*_n$ if and only if the function $F: B \times [0, \infty) \to \mathbb{C}^n$ given by $F(z, t) = e^t F(z)$ is a Loewner chain.

We can now prove the following.

Theorem 2.2. If $\Phi: \mathcal{L}S_1 \to \mathcal{L}S_n$ is a Loewner chain preserving extension operator, then $\Phi(S_1) \subseteq S^*_n$ and $\Phi(S^*_1) \subseteq S^*_n$.

Proof. If $f \in S_1$, then $f$ can be embedded as the first element of a Loewner chain $f: \Delta \times [0, \infty) \to \mathbb{C}$. It follows that $F$ given in (2.1) is a Loewner chain. Now $S_1$ is compact and \( \{ e^{-t} f(\cdot, t) : t \geq 0 \} \subseteq S_1 \). It follows, by the continuity of $\Phi$, that

\[
\{ e^{-t} F(\cdot, t) : t \geq 0 \} = \{ \Phi(e^{-t} f(\cdot, t)) : t \geq 0 \}
\]

lies in the compact set $\Phi(S_1)$ and hence is a normal family. Thus $\Phi(f) = F(\cdot, 0) \in S^*_n$.

The relation $\Phi(S^*_1) \subseteq S^*_n$ follows as above using Theorem 2.1. $\square$

The Roper–Suffridge extension operator is defined by

\[
[\Phi_{0,1/2}(f)](z) = \left( f(z_1), \sqrt{f'(z_1)} \hat{z} \right), \quad f \in \mathcal{L}S_1, \ z \in B. \tag{2.2}
\]

(Since $f'(\Delta)$ is a simply connected set that does not contain 0, a branch of $\sqrt{f'(z)}$ may be chosen such that $\sqrt{f'(0)} = 1$.) It was shown by Roper and Suffridge [16] that $\Phi_{0,1/2}(\mathcal{K}_1) \subseteq \mathcal{K}_n$. Later, Graham and Kohr [5] proved that $\Phi_{0,1/2}(S_1^1) \subseteq S_n^*$.

The author showed [9] that analogous results hold for the extension operator

\[
[\Phi_{Q,1/2}(f)](z) = \left( f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)} \hat{z} \right), \quad f \in \mathcal{L}S_1, \ z \in B, \tag{2.3}
\]

for certain $Q \in \mathcal{P}_2(n - 1)$. In particular, $\Phi_{Q,1/2}(\mathcal{K}_1) \subseteq \mathcal{K}_n$ if and only if $\|Q\| \leq 1/2$ and $\Phi_{Q,1/2}(S_1^1) \subseteq S_n^*$ if and only if $\|Q\| \leq 1/4$.

Graham, G. Kohr, and M. Kohr [6] and Graham, Hamada, Kohr, and Suffridge [3] considered the application of Loewner chains to the analysis of extension operators. They studied extension operators of the form

\[
[\Psi_{\alpha,\beta}(f)](z) = \left( f(z_1), \left[ \frac{f(z_1)}{z_1} \right]^\alpha [f'(z_1)]^\beta \hat{z} \right), \quad f \in \mathcal{L}S_1, \ z \in B, \tag{2.4}
\]

where $\alpha, \beta \geq 0$. (In [6], the operator $\Psi_{0,\beta}$ was studied. As above, branches of $\zeta \mapsto [f(\zeta)/\zeta]^\alpha$ and $\zeta \mapsto [f'(\zeta)]^\beta$ can be chosen to have value 1 at $\zeta = 0$.) Using the above definition, their results can be rephrased to say that $\Psi_{\alpha,\beta}$ is a Loewner chain preserving extension operator if $\alpha \in [0, 1]$, $\beta \in [0, 1/2]$, and $\alpha + \beta \leq 1$. It then follows that for such $\alpha$ and $\beta$, $\Psi_{\alpha,\beta}(S_1^1) \subseteq S_n^*$ and $\Psi_{\alpha,\beta}(S_1^*) \subseteq S_n^*$.

We now introduce our primary object of study. Let $G \in H(\mathbb{C}^{n-1}, \mathbb{C})$ be such that $G(0) = 0$ and $DG(0) = 0$, and let $\beta \geq 0$. For $f \in \mathcal{L}S_1$, define $[\Phi_{G,\beta}(f)] \in H(B(\mathbb{C}^n))$ by

\[
[\Phi_{G,\beta}(f)](z) = \left( f(z_1) + G([f'(z_1)]^\beta \hat{z}), [f'(z_1)]^\beta \hat{z} \right), \quad z \in B. \tag{2.5}
\]

(Again, we may choose a branch of $[f'(\cdot)]^\beta$ such that $[f'(0)]^\beta = 1$.) A simple calculation verifies that $f$ univalent on an open set $U \subseteq \Delta$ implies that $\Phi_{G,\beta}(f)$ is biholomorphic on the
set \((U \times \mathbb{C}^{n-1}) \cap B\). Hence \(\Phi_{G,\beta} : \mathcal{L}S_1 \to \mathcal{L}S_n\), and, in particular, \(\Phi_{G,\beta}(S_1) \subseteq S_n\). It follows from Vitali’s theorem [4] that \(\Phi_{G,\beta}\) is continuous with respect to the natural topologies of \(\mathcal{L}S_1\) and \(\mathcal{L}S_n\). For, if \(\{f_m\}_{m=1}^\infty \subseteq \mathcal{L}S_1\) converges locally uniformly to \(f \in \mathcal{L}S_1\), then \(f_m' \to f'\) locally uniformly and \(\{f_m\}\) and \(\{f_m'\}\) are locally uniformly bounded. Therefore \(\{\Phi_{G,\beta}(f_m)\}\) is a locally uniformly bounded sequence that converges pointwise to \(\Phi_{G,\beta}(f)\), as is sufficient. We now see that \(\Phi_{G,\beta}\) is an extension operator. In Sections 3 and 4, we will study conditions under which \(\Phi_{G,\beta}\) is Loewner chain preserving. Amongst other results, we will see that the maximum degree of terms appearing the homogeneous expansion of \(G\) is a function of \(\beta\).

By way of Theorem 2.2, we see that any extension operator that is Loewner chain preserving also preserves starlike mappings. For completeness, we give the following theorem that shows some of the limitations present in the study of convex mappings with regard to extension operators.

**Theorem 2.3.** Let \(\Phi : \mathcal{L}S_1 \to \mathcal{L}S_n\) be an extension operator, and let \(\varphi_\theta \in \mathcal{K}_1\), \(\theta \in \mathbb{R}\), be the half-plane mapping

\[
\varphi_\theta(\zeta) = \frac{\zeta}{1 - e^{-i\theta}\zeta}, \quad \zeta \in \Delta. \tag{2.6}
\]

If \(\Phi(\varphi_\theta) \in \mathcal{K}_n\), then there is some \(Q \in \mathcal{P}_2(n - 1)\) such that \(\|Q\| \leq 1/2\) and \(\Phi(\varphi_\theta) = \Phi_{Q,1/2}(\varphi_\theta)\).

**Proof.** Let \(F \in \mathcal{K}_n\) be the rotation of \(\Phi(\varphi_\theta)\) given by

\[
F(z) = e^{-i\theta}[\Phi(\varphi_\theta)](e^{i\theta}z), \quad z \in B.
\]

Then

\[
F(z_1e_1) = e^{-i\theta}[\Phi(\varphi_\theta)](e^{i\theta}z_1e_1) = e^{-i\theta}\varphi_\theta(e^{i\theta}z_1)e_1 = \frac{z_1}{1 - z_1}e_1, \quad z_1 \in \Delta.
\]

We see that \(F(B)\) contains the line \(\{ite_1 : t \in \mathbb{R}\}\), and hence, by convexity (see [12]), is equal to the union of lines parallel to this line. Since

\[
\lim_{t \to \infty} F^{-1}(F(\cdot) + ite_1), \quad \lim_{t \to -\infty} F^{-1}(F(\cdot) + ite_1)
\]

are each equal to the constant function taking the value \(e_1\), the convergence being locally uniform on \(B\) (see [12] again), we conclude from results in [11] and [10] that there is some \(P \in \mathcal{P}_2(n - 1)\) with \(\|P\| \leq 1/2\) such that

\[
F(z) = \left(\frac{z_1}{1 - z_1} + P\left(\frac{z}{1 - z_1}\right), \frac{z}{1 - z_1}\right), \quad z \in B.
\]

Since \([\Phi(\varphi_\theta)](z) = e^{i\theta}F(e^{-i\theta}z)\) for \(z \in B\), we have that \(\Phi(\varphi_\theta) = \Phi_{Q,1/2}(\varphi_\theta)\), where \(Q = e^{-i\theta}P\).

The mappings \(\varphi_\theta\) above are the extreme points of the family \(\mathcal{K}_1\) (see [1]), and we now see that the only way to extend an extreme point of \(\mathcal{K}_1\) to a mapping in \(\mathcal{K}_n\) is using an extension operator of the form \(\Phi_{Q,1/2}\) with \(Q \in \mathcal{P}_2(n - 1)\) such that \(\|Q\| \leq 1/2\). This is
not to say that the only extension operators taking $\mathcal{K}_1$ into $\mathcal{K}_n$ are of the form $\Phi_{Q,1/2}$. Certainly the extension operator

$$[\Phi(f)](z) = \left( f(z_1) + \frac{f''(0)}{2} f'(z_1)Q(\zeta), \sqrt{f'(z_1)} \zeta \right), \quad f \in \mathcal{L}_1, \ z \in B,$$

where $Q \in \mathcal{P}_2(n-1)$, $\|Q\| \leq 1/2$, satisfies $\Phi(\mathcal{K}_1) \subseteq \mathcal{K}_n$ since $|f''(0)| \leq 2$ must be true of all $f \in \mathcal{K}_1$.

### 3 Loewner chains and the operator $\Phi_{G,\beta}$

We begin with a helpful lemma.

**Lemma 3.1.** Let $f \in \mathcal{S}_1$ and $\alpha \geq 2$. Then

$$\left| \frac{1 - |\zeta|^2}{\alpha} \frac{f''(\zeta)}{f'(\zeta)} - \zeta \right| \leq \frac{4 + (\alpha - 2)|\zeta|}{\alpha}, \quad \zeta \in \Delta. \quad (3.1)$$

**Proof.** Fix $\zeta \in \Delta$, and set

$$A = \frac{1 - |\zeta|^2}{2} \frac{f''(\zeta)}{f'(\zeta)}.$$

Let $g \in \mathcal{S}_1$ be the Koebe transform of $f$ with respect to the disk automorphism

$$\varphi(w) = \frac{w + \zeta}{1 + \zeta w}, \quad w \in \Delta.$$

Then

$$g(w) = \frac{f(\varphi(w)) - f(\varphi(0))}{f'(\varphi(0))\varphi'(0)} = w + (A - \zeta)w^2 + O(|w|^3), \quad w \in \Delta.$$

Let $\gamma = A - \zeta$. It follows that $|\gamma| \leq 2$. With $\beta = \alpha/2$, we have

$$\left| \frac{A}{\beta} - \zeta \right| = \left| \frac{\gamma - (\beta - 1)\zeta}{\beta} \right| \leq \frac{|\gamma| + (\beta - 1)|\zeta|}{\beta},$$

giving the result. \qed

Before stating the main result of this article, let us recall some facts about homogeneous polynomials. If $P \in \mathcal{P}_m(n)$ for some $m \in \mathbb{N}$, then there is a symmetric $m$-linear functional $L : \prod_{j=1}^m \mathbb{C}^n \to \mathbb{C}$ such that

$$P(z) = L(z, \ldots, z), \quad z \in \mathbb{C}^n.$$ 

It follows from a simple calculation that for $z \in \mathbb{C}^n$,

$$DP(z) = mL(z, \ldots, z, \cdot).$$

(This is a linear functional on $\mathbb{C}^n$, which can be thought of as a $1 \times n$ matrix.) Hence

$$DP(z)z = mP(z).$$

The following is our main theorem.
Theorem 3.2. Let $\beta \in [0, 1/2]$, and let $G \in H(\mathbb{C}^{n-1}, \mathbb{C})$ have homogeneous expansion $G = \sum_{j=0}^{\infty} P_j$, where $P_j \in \mathcal{P}_j(n - 1)$ for each $j = 2, 3, \ldots$. Suppose that $P_j = 0$ for all $j < 4/(2\beta + 1)$ and, if $\beta > 1/6$, that $P_j = 0$ for all $j > 4/(6\beta - 1)$. There exists a constant

$$C_\beta \geq \inf_{0 < x < 1} \frac{2(1 - \beta)x^2 + 2(1 - 2\beta)x + 1}{4\beta + (1 - 2\beta)x} \geq \frac{1}{1 + 2\beta}$$

such that $\Phi_{G,\beta}$ is a Loewner chain preserving extension operator provided that

$$\sum_{j=1}^{[1/\beta]} (j - 1)2^{(1-\beta)j/2+4}\|P_j\| + \sum_{j=[1/\beta]+1}^{\infty} (j - 1)2^{(2\beta+1)j/2}\|P_j\| \leq C_\beta. \quad (3.3)$$

In Section 4, we will see that the constraint placed on the degree of the terms of $G$ when $\beta \in [1/6, 1/2]$ is necessary, even in a more general setting. It is worth noting that the infimum in (3.2) can be solved for using calculus, but the solution is unappealing, and so we omit it.

Proof. Let $f : \Delta \times [0, \infty) \to \mathbb{C}$ be a Loewner chain, and define $F : B \times [0, \infty) \to \mathbb{C}^n$ by

$$F(z, t) = \left( f(z_1, t) + e^{t}\Phi_{G,\beta}(e^{-t}\Phi_{G,\beta}(f(z_1, t)\alpha), e^{(1-\beta)t}\Phi_{G,\beta}(f(z_1, t)^\beta\alpha)), \quad z \in B, \ t \geq 0. \quad (3.4) \right)$$

We must show that $F$ is a Loewner chain and will apply Theorem 1.1 to do so. Clearly $F(\cdot, t) \in H(B, \mathbb{C}^n), F(0, t) = 0$, and $DF(0, t) = e^tI$ for all $t \geq 0$.

To see that $F(z, t)$ is a locally absolutely continuous function of $t$ locally uniformly with respect to $z$, it suffices to show that $F$ is a Lipschitz continuous function of $t \in [0, T]$ uniformly with respect to $z \in r\overline{B}$ for $T > 0$ and $r \in (0, 1)$. Let $\rho \in (r, 1)$. We know that $|f(z_1, s) - f(z_1, t)| \leq M|s - t|$ for some constant $M > 0$ and all $s, t \in [0, T]$ and $z_1 \in \rho\overline{\Delta}$. But then $f'(z_1, t)$ is a Lipschitz continuous function of $t \in [0, T]$ uniformly with respect to $z_1 \in r\overline{\Delta}$, as seen through the calculation

$$|f'(z_1, t) - f'(z_1, s)| \leq \frac{1}{2\pi} \int_{\partial(\rho\overline{\Delta})} \frac{|f(\zeta, t) - f(\zeta, s)|}{|\zeta - z_1|^2} |d\zeta| \leq \frac{M}{(\rho - r)^2}|t - s|,$$

for $s, t \in [0, T]$ and $z_1 \in r\overline{\Delta}$. One can easily verify that $f'$ is continuous on $r\overline{\Delta} \times [0, T]$, and hence $f'(r\overline{\Delta} \times [0, T])$ is a compact subset of $\mathbb{C} \setminus \{0\}$. It follows that $[f'(z_1, t)]^\beta$ is also a Lipschitz continuous function of $t \in [0, T]$ uniformly with respect to $z_1 \in r\overline{\Delta}$. Since sums and products of Lipschitz continuous functions are Lipschitz continuous, the result is evident with the observation that $G$ is Lipschitz continuous on $\sigma B_{n-1}$ for any $\sigma > 0$, which follows from the calculation

$$|G(v) - G(u)| \leq \int_0^1 \|DG((1 - \tau)u + \tau v)\||v - u||d\tau$$

for $u, v \in \sigma B_{n-1}$ and that $\|DG(\cdot)\|$ is bounded on $\sigma B_{n-1}$.

Since $\{e^{-t}f(\cdot, t) : t \geq 0\} \subseteq \mathcal{S}_1$ and $\mathcal{S}_1$ is a compact family, there is an increasing sequence $\{t_m\}_{m=1}^{\infty} \subseteq [0, \infty)$ such that $t_m \to \infty$ and a function $g \in \mathcal{S}_1$ such that $e^{-t_m}f(\cdot, t_m) \to g$ locally uniformly. Observe that

$$e^{-t_m}F(\cdot, t_m) = \Phi_{G,\beta}(e^{-t_m}f(\cdot, t_m)).$$
By continuity of $\Phi_{G, \beta}$, this sequence converges locally uniformly to the function $\Phi_{G, \beta}(g)$.

For $z \in B$ and $t \geq 0$ we calculate the block matrix,

$$DF(z, t) =
\begin{bmatrix}
  f'(z, t) + \sum_{j=2}^{\infty} \beta_{j} e^{(1-\beta_{j})t}[f'(z, t)]^{\beta_{j}-1} f''(z, t)P_{j}(\hat{z}) + \sum_{j=2}^{\infty} e^{(1-\beta_{j})t}[f'(z, t)]^{\beta_{j}} DP_{j}(\hat{z})
  \\
  \beta e^{(1-\beta)t}[f'(z, t)]^{\beta-1} f''(z, t) \hat{z}
  \\
  e^{(1-\beta)t}[f'(z, t)]^{\beta} I_{n-1}
\end{bmatrix},$$

where $I_{n-1}$ is the identity operator on $\mathbb{C}^{n-1}$.

There exists a function $p: \Delta \times [0, \infty) \rightarrow \mathbb{C}$ such that for all $z_{1} \in \Delta$ and $t \geq 0$, $p(\cdot, t)$ is analytic, $p(z_{1}, \cdot)$ is measurable, $p(0, t) = 1$, $\text{Re} p(z_{1}, t) > 0$, and

$$\frac{\partial f}{\partial t}(z_{1}, t) = z_{1} f'(z_{1}, t)p(z_{1}, t), \quad z_{1} \in \Delta, \text{ a.e. } t \geq 0. \quad (3.5)$$

(The omitted set of measure zero in $[0, \infty)$ is independent of $z_{1}$.)

Let $t_{0} \geq 0$ be such that $[\partial f / \partial t](\cdot, t_{0})$ exists, and let $\{t_{k}\}_{k=1}^{\infty} \subseteq [0, t_{0} + 1]$ be an arbitrary sequence converging to $t_{0}$. Write

$$\frac{\partial f}{\partial t}(z, t_{0}) = \lim_{k \to \infty} \frac{f(z, t_{k}) - f(z, t_{0})}{t_{k} - t_{0}}, \quad z_{1} \in \Delta. \quad (3.6)$$

Because $f(z_{1}, t)$ is a Lipschitz continuous function of $t \in [0, t_{0} + 1]$ locally uniformly with respect to $z_{1} \in \Delta$, the quotient within the limit in (3.6) is locally uniformly bounded in $\Delta$ and hence the convergence is locally uniform on $\Delta$ by Vitali's theorem [2]. Differentiation in $z_{1}$ may therefore pass through the limit, which justifies the reversal in the order of differentiation in the calculation

$$\frac{\partial f'}{\partial t}(z_{1}, t) = f'(z_{1}, t)p(z_{1}, t) + z_{1} f''(z_{1}, t)p(z_{1}, t) + z_{1} f'(z_{1}, t)p'(z_{1}, t), \quad z_{1} \in \Delta, \text{ a.e. } t \geq 0. \quad (3.7)$$

With the substitutions (3.5) and (3.7), one can directly calculate that

$$\frac{\partial F}{\partial t}(z, t) = DF(z, t)h(z, t), \quad z \in B, \text{ a.e. } t \geq 0,$$

where $h: B \times [0, \infty) \rightarrow \mathbb{C}^{n}$ is defined by

$$h(z, t) = \left( z_{1}p(z_{1}, t) - \sum_{j=2}^{\infty} (j - 1)e^{(1-\beta_{j})t}[f'(z_{1}, t)]^{\beta_{j}-1}P_{j}(\hat{z}),
\begin{bmatrix}
  1 - \beta + \beta p(z_{1}, t) + \beta z_{1}p'(z_{1}, t) + \beta \sum_{j=2}^{\infty} (j - 1)e^{(1-\beta_{j})t}[f'(z_{1}, t)]^{\beta_{j}-2}f''(z_{1}, t)P_{j}(\hat{z})
\end{bmatrix} \hat{z} \right). \quad (3.8)$$

It is clear that for all $t \geq 0$, $h(\cdot, t)$ is holomorphic, $h(0, t) = 0$, and $Dh(0, t) = I$. That $h(z, \cdot)$ is measurable for $z \in B$ follows from (3.7), noting that, for all $z_{1} \in \Delta$, $f'(z_{1}, \cdot)$ and $f''(z_{1}, \cdot)$ are locally absolutely continuous and hence $[\partial f'/\partial t](z_{1}, \cdot)$ is measurable.
It remains to show that \( \text{Re}(h(z, t), z) > 0 \) for all \( z \in B \setminus \{0\} \) and \( t \geq 0 \). If \( \hat{z} = 0 \), then
\[
\text{Re}(h(z, t), z) = |z|^2 \text{Re}p(z_1, t) > 0
\]
for \( t \geq 0 \). Now consider \( z \in B \) such that \( \hat{z} \neq 0 \). For fixed \( t \geq 0 \), \( h(\cdot, t) \) can be extended to be holomorphic in a neighborhood of each \( z \in \overline{B} \) such that \( \hat{z} \neq 0 \). Write \( z = \lambda Z \) for some fixed \( Z \in \partial B \) such that \( \hat{Z} \neq 0 \) and \( \lambda \in \Delta \setminus \{0\} \). Then \( \text{Re}(h(z, t), z) \geq 0 \) if and only if
\[
\text{Re}\left( \frac{h(\lambda Z, t)}{\lambda}, Z \right) \geq 0.
\]
The left-hand side of the above can be seen to be the real part of a nonconstant analytic function of the complex variable \( \lambda \in \overline{\Delta} \), and is hence harmonic. By the minimum principle for harmonic functions, it will attain its minimum for some \( \lambda \in \partial \Delta \), and hence for \( z \in \partial B \).

It therefore suffices to prove \( \text{Re}(h(z, t), z) \geq 0 \) for all \( z \in \partial B \) with \( \hat{z} \neq 0 \) and all \( t \geq 0 \).

Fix such \( z \) and \( t \). Observe that
\[
\text{Re}(h(z, t), z) = (1 - \beta)\|z\|^2 + (|z|^2 + \beta\|\hat{z}\|^2) \text{Re}p(z_1, t) + \beta\|\hat{z}\|^2 \text{Re}[z_1p'(z_1, t)]
\]
\[
+ \text{Re}\left( \beta\|\hat{z}\|^2 \frac{f''(z_1, t)}{f'(z_1, t)} - \bar{z}_1 \right) \sum_{j=2}^{\infty} (j - 1)[e^{-t}f'(z_1, t)]^{\beta j - 1}P_{2j}(z) \tag{3.9}
\]
\[
\text{It is well-known [4, line (2.1.6)] that}
\]
\[
|p'(z_1, t)| \leq \frac{2 \text{Re}p(z_1, t)}{1 - |z|^2} \tag{3.10}
\]
\[
\text{Therefore to prove } \text{Re}(h(z, t), z) \geq 0 \text{ it suffices to show that}
\]
\[
\left| \beta\|\hat{z}\|^2 \frac{f''(z_1, t)}{f'(z_1, t)} - \bar{z}_1 \right| \sum_{j=2}^{\infty} (j - 1)[e^{-t}f'(z_1, t)]^{\beta j - 1}P_{2j}\|\hat{z}\|^j \leq (1 - \beta)\|\hat{z}\|^2 + ((1 - \beta)|z|^2 - 2\beta|z_1| + \beta) \text{Re}p(z_1, t). \tag{3.11}
\]
\[
\text{Since } e^{-t}f(\cdot, t) \in S_1 \text{ for all } t \geq 0, \text{ we use Lemma 3.1 to see that}
\]
\[
\left| \beta\|\hat{z}\|^2 \frac{f''(z_1, t)}{f'(z_1, t)} - \bar{z}_1 \right| \leq 4\beta + (1 - 2\beta)|z_1|.
\]
\[
\text{Since } (1 - \beta)|z|^2 - 2\beta|z_1| + \beta \geq \beta(|z|^2 - 1)^2 \geq 0, \text{ we may also use the well-known estimate}
\]
\[
\text{Re}p(z_1, t) \geq \frac{1 - |z_1|^2}{1 + |z_1|} \tag{3.12}
\]
\[
\text{With these, we apply the distortion bounds}
\]
\[
\frac{1 - |z_1|}{(1 + |z_1|)^3} \leq |e^{-t}f'(z_1, t)| \leq \frac{1 + |z_1|}{(1 - |z|^2)^3},
\]
\[
\text{The left-hand side of the above can be seen to be the real part of a nonconstant analytic function of the complex variable } \lambda \in \overline{\Delta}, \text{ and is hence harmonic. By the minimum principle for harmonic functions, it will attain its minimum for some } \lambda \in \partial \Delta, \text{ and hence for } z \in \partial B.
\]

It therefore suffices to prove \( \text{Re}(h(z, t), z) \geq 0 \) for all \( z \in \partial B \) with \( \hat{z} \neq 0 \) and all \( t \geq 0 \).
the lower when $j \leq 1/\beta$ and the upper when $j > 1/\beta$, to see that it suffices to show
\[
\sum_{j=2}^{[1/\beta]} (j - 1)\|P_j\|(1 + |z_1|)(1-6\beta)j/2 + (1 - |z_1|)(2\beta+1)j/2 - 2
\]
\[
+ \sum_{j=[1/\beta]+1}^{\infty} (j - 1)\|P_j\|(1 + |z_1|)(2\beta+1)j/2(1 - |z_1|)(1-6\beta)j/2 + 2
\]
\[
\leq \frac{2(1 - \beta)|z_1|^2 + 2(1 - 2\beta)|z_1| + 1}{4\beta + (1 - 2\beta)|z_1|}.
\]
In the series on the left-hand side, the exponent on $1 + |z_1|$ is positive for all $j$, and $\|P_j\| = 0$ if $j$ is such that the exponent on $1 - |z_1|$ is negative. The result follows.

To see that $C_\beta < \infty$ for any $\beta \in [0, 1/2]$, simply consider the Loewner chain $f: \Delta \times [0, \infty) \to \mathbb{C}$ given by $f(\zeta, t) = e^t \zeta$, and let $G = Q \in \mathcal{P}_2(n - 1)$. Now $F: B \times [0, \infty) \to \mathbb{C}^n$ given in (3.4) is then
\[
F(z, t) = e^t(z_1 + Q(\hat{z}), \hat{z}), \quad z \in B, \ t \geq 0.
\]
If $F$ is a Loewner chain, then the function $F(z) = (z_1 + Q(\hat{z}), \hat{z})$ is starlike by Theorem 2.1. This cannot be true for arbitrarily large $\|Q\|$, meaning $C_\beta$ must be finite.

Let us now consider the special case in which $\beta = 1/m$ for some $m \in \mathbb{N}$, $m \geq 2$, and $G = P \in \mathcal{P}_m(n - 1)$. We can improve Theorem 3.2 as follows.

**Corollary 3.3.** Let $m \in \mathbb{N}$, $m \geq 2$, and $P \in \mathcal{P}_m(n - 1)$. Then $\Phi_{P,1/m}$ is a Loewner chain preserving extension operator provided that
\[
\|P\| \leq \inf_{0 < x < 1} \frac{2(m - 1)x^2 + 2(m - 2)x + m}{(m - 1)[4 + (m - 2)x](1 - x)(m - 2)/2(1 + x)(m + 2)/2}. \quad (3.13)
\]

The infimum in (3.13) can be solved for – it is the root of a 4th degree polynomial whose coefficients depend upon $m$.

**Proof.** Let $f: \Delta \times [0, \infty) \to \mathbb{C}$ be a Loewner chain, and let $F: B \times [0, \infty) \to \mathbb{C}^n$ be given by
\[
F(z, t) = \left( f(z_1, t) + f'(z_1, t)P(\hat{z}), e^{(m-1)t/m}[f'(z_1, t)]^{1/m} \hat{z} \right). \quad (3.14)
\]
Consider the proof of Theorem 3.2. Under our current hypotheses, (3.11) becomes
\[
(m - 1)[4 + (m - 2)|z_1||P|(1 - |z_1|^2)^{m/2}
\]
\[
\leq (m - 1)(1 - |z_1|^2) + [(m - 1)|z_1|^2 - 2|z_1| + 1]^2 - |z_1| - 1 + |z_1|.
\]
Solve for $\|P\|$ to see that $F$ is a Loewner chain if $\|P\|$ satisfies (3.13). \qed
Although it is an improvement over (3.2), it appears that the bound (3.13) is not tight if \( m \geq 3 \). For, if we choose \( f : \Delta \times [0, \infty) \to \mathbb{C} \) to be the Loewner chain associated to the Koebe function

\[
f(\zeta, t) = \frac{e^t \zeta}{1 - \zeta^2}, \quad \zeta \in \Delta, \ t \geq 0,
\]

then the bounds (3.1), (3.10), and (3.12) are each separately tight for certain \( z_1 \in \Delta \) and all \( t \geq 0 \), but not for the same values of \( z_1 \). Whether there is a way to improve upon this is not clear. However, if \( m = 2 \), we do have the following theorem. We note that this result was developed independently by Kohr in a recent paper [8].

**Theorem 3.4.** Let \( Q \in \mathcal{P}_2(n - 1) \). Then \( \Phi_{Q,1/2} \) is a Loewner chain preserving extension operator if and only if \( \|Q\| \leq 1/4 \).

**Proof.** We apply Corollary 3.3. With \( m = 2 \), (3.13) becomes

\[
\|Q\| \leq \inf_{0 < x < 1} \frac{x^2 + 1}{2(x+1)^2}.
\]

The expression within the infimum is clearly a decreasing function of \( x \), and hence \( \Phi_{Q,1/2} \) is a Loewner chain preserving extension operator if \( \|Q\| \leq 1/4 \).

Assume \( f \) is as given in (3.15), and consider the proof of Theorem 3.2. It follows that

\[
p(z_1, t) = \frac{1 - z_1}{1 + z_1}, \quad z_1 \in \Delta, \ t \geq 0.
\]

Let \( z_1 = r \in (0, 1) \), and let \( \hat{z} \in B_{n-1} \) be such that \( \|\hat{z}\| = \sqrt{1 - r^2} \) and \( Q(\hat{z}) = -(1 - r^2)\|Q\| \).

One may then see that (3.9) becomes

\[
2 \text{Re} \langle h(z, t), z \rangle = \frac{(1 - r)^3}{1 + r} + 1 - r^2 - 4(1 - r^2)\|Q\|
\]

\[
= (1 - r^2) \left[ \left( \frac{1 - r}{1 + r} \right)^2 + 1 - 4\|Q\| \right].
\]

For \( \text{Re} \langle h(z, t), z \rangle \geq 0 \), we require

\[
4\|Q\| \leq \left( \frac{1 - r}{1 + r} \right)^2 + 1, \quad 0 < r < 1.
\]

It immediately follows that \( \|Q\| \leq 1/4 \) is necessary.

The following important corollary to Theorem 3.2 follows from Theorem 2.2.

**Corollary 3.5.** Let \( \beta \in [0, 1/2] \) and \( G = \sum_{j=2}^{\infty} P_j \in H(\mathbb{C}^{n-1}, \mathbb{C}) \) satisfy the hypotheses of Theorem 3.2. Then \( \Phi_{G,\beta}(S_1) \subseteq S_n^* \) and \( \Phi_{G,\beta}(S_1^*) \subseteq S_n^* \). If \( \beta = 1/2 \) and \( G = Q \in \mathcal{P}_2(n - 1) \), then \( \Phi_{Q,1/2}(S_1) \subseteq S_n^* \) and \( \Phi_{Q,1/2}(S_1^*) \subseteq S_n^* \) each occur if and only if \( \|Q\| \leq 1/4 \).

That \( \Phi_{Q,1/2}(S_1^*) \subseteq S_n^* \) if and only if \( \|Q\| \leq 1/4 \) was proved using a different method in [9].

Consider Corollary 3.3. We provide the following estimates on the size of \( \|P\| \) that are more appealing, although less precise, than the condition (3.13).
Corollary 3.6. Let \( m \) and \( P \) be as in the hypotheses of Corollary 3.3. Then \( \Phi_{P,1/m} \) is a Loewner chain preserving extension operator provided that

\[
\|P\| \leq \frac{(m^2 + 2m - 4)(m - 1)^{m-2}}{(m + 2)(2m + (m - 2)\sqrt{m + 3})(m^2 - 3m - 6 + 4\sqrt{m + 3})(m-2)/2} \tag{3.16}
\]

or

\[
\|P\| \leq \frac{1}{m + 2}. \tag{3.17}
\]

Although the bound (3.17) is much more pleasant than the bound (3.16) it is also far less accurate. Note that if \( m = 2 \), then both (3.16) and (3.17) reduce to \( \|P\| = 4 \), which is the tight bound established in Theorem 3.4.

Proof. Define the functions \( \varphi : [0, 1] \to \mathbb{R} \) and \( \psi : [0, 1] \to \mathbb{R} \) by

\[
\varphi(x) = \frac{2(m - 1)x^2 + 2(m - 2)x + m}{(x + 1)^2}, \quad \psi(x) = \frac{1}{[4 + (m - 2)x](1 - x^2)(m-2)/2}.
\]

Now (3.13) can be rewritten

\[
\|P\| \leq \inf_{0 < x < 1} \frac{\varphi(x)\psi(x)}{m - 1}.
\]

Using elementary calculus, one may verify that \( \varphi \) attains its minimum at \( x = 2/m \) and \( \psi \) attains its minimum at \( x = (\sqrt{m + 3} - 2)/(m - 1) \). Therefore \( \Phi_{P,1/m} \) is a Loewner chain preserving extension operator provided that

\[
\|P\| \leq \frac{1}{m - 1} \varphi \left( \frac{2}{m} \right) \psi \left( \frac{\sqrt{m + 3} - 2}{m - 1} \right),
\]

which gives (3.16).

To prove (3.17), let \( f : \Delta \times [0, \infty) \to \mathbb{C} \) be a Loewner chain, and let \( F : B \times [0, \infty) \to \mathbb{C}^n \) be given by (3.14). We consider the proof of Theorem 3.2, particularly line (3.9). If \( \text{Re}(h(z, t), z) \geq 0 \) for all \( z \in \Delta \), then \( F \) is a Loewner chain. Use Lemma 3.1 to see that

\[
\left| \frac{\|\hat{z}\|^2 f''(z_1, t)}{f'(z_1, t)} - m\bar{z}_1 \right| \leq m + 2, \quad z \in \partial B, \ \hat{z} \neq 0.
\]

Use (3.10) and that \( \text{Re}(p(z_1, t)) > 0 \) for all \( z_1 \in \Delta \) and \( t \geq 0 \) to see that \( \text{Re}(h(z, t), z) \geq 0 \) provided that

\[
(m + 2)\|P\|\|\hat{z}\|^m \leq \|\hat{z}\|^2, \quad \hat{z} \in \overline{B}_{n-1}.
\]

This gives (3.17).

So far, we have considered the upper bound on \( \|P\| \) so that \( \Phi_{P,1/m}(S_1) \subseteq S^0_{n} \) or \( \Phi_{P,1/m}(S^*_1) \subseteq S^*_n \). However if a particular function \( f \in S_1 \) (respectively \( f \in S^*_1 \)) is specified, the bound on \( \|P\| \) such that \( \Phi_{P,1/m}(f) \in S^0_{n} \) (respectively \( \Phi_{P,1/m}(f) \in S^*_n \)) may be larger, as is seen in the following example.
Example 3.7. Let $m \in \mathbb{N}$, $m \geq 2$, $\beta = 1/m$, $G = P \in \mathcal{P}_m(n-1)$, and consider $f(\zeta) = \zeta$ in the proof of Theorem 3.2. Since $f \in \mathcal{S}^*_1$, we have that $f(\zeta,t) = e^t \zeta$ is a Loewner chain, and clearly $p(z_1,t) = 1$ for all $z_1 \in \Delta$ and $t \geq 0$. Therefore (3.9) becomes

$$\text{Re} \langle h(z,t), z \rangle = 1 - (m-1) \text{Re}(\zeta_1 P(\zeta)).$$

For any $z_1 \in \Delta$, $\tilde{z}$ can be chosen such that $\|\tilde{z}\| = \sqrt{1 - |z_1|^2}$ and $\zeta_1 P(\zeta) = |z_1||\tilde{z}|^m P$. Therefore, $\Phi_{P,1/m}(f) \in \mathcal{S}^*_n$ (equivalently $\Phi_{P,1/m}(f) \in \mathcal{S}^0_n$) if and only if

$$(m-1)|z_1|(1 - |z_1|^2)^{m/2} \|P\| \leq 1,$$  

$z \in \partial B$.

Simple calculus reveals that

$$\max_{0 \leq x \leq 1} x(1 - x)^{m/2} = \frac{m^{m/2}}{(m+1)(m+1/2)}.$$

It is therefore necessary and sufficient that

$$\|P\| \leq \frac{(m+1)(m+1/2)}{(m-1)m^{m/2}}.$$

This upper bound on $\|P\|$ is significantly larger than the upper bound (3.13).

4 Constraints on mappings in $\mathcal{S}^0_n$

Let $\beta \in [0,1/2]$. In the hypotheses of Theorem 3.2, as well as in Corollary 3.5, we require that $G$ be a polynomial of limited degree if $\beta > 1/6$, while we allow $G$ to have terms of arbitrarily large degree if $\beta \leq 1/6$. When considering whether $\Phi_{G,\beta}(f) \in \mathcal{S}^0_n$ for a particular $f \in \mathcal{S}_1$, we need only that $\Phi_{G,\beta}(f)$ be embedded as the first element of some Loewner chain, not necessarily the one given by (3.4). As the following theorem shows, if $f$ is the Koebe function, then the restriction on the degree of the terms of $G$ specified in Theorem 3.2 and its corollaries is indeed necessary for $\Phi_{G,\beta}(f)$ to be in $\mathcal{S}^0_n$.

Theorem 4.1. Let $\beta \in (1/6,1/2]$, $f \in \mathcal{S}_1$ be the Koebe function

$$f(\zeta) = \frac{\zeta}{(1-\zeta)^2}, \quad \zeta \in \Delta,$$

and $F \in \mathcal{S}_n$ be given by

$$F(z) = \left( f(z_1), [f'(z_1)]^\beta \tilde{z} \right) + G([f'(z_1)]^\beta \tilde{z}), \quad z \in B,$$

where $G \in H(\mathbb{C}^{n-1}, \mathbb{C}^n)$ satisfies $G(0) = 0$ and $DG(0) = 0$. If $F \in \mathcal{S}^0_n$, then $G$ must be a polynomial of degree at most $\lfloor 4/(6\beta - 1) \rfloor$.

Of course if $\beta \leq 1/6$, then Corollary 3.5 shows that there is no limit on the degree of the terms of $G$. 

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Let us fix \( z_1 = r \in [0, 1) \). For \( k = 2, \ldots, n \), let \( \rho_k > 0 \) be such that \( \sum_{k=2}^{n} \rho_k^2 = 1/4 \), and assume \( |z| = \rho_k \sqrt{1 - r^2} \). It follows that \( \|z\| \leq (1 + r)/2 \). We expand \( G \), using multi-indices, as

\[
G(w) = \sum_{\alpha \in \mathbb{N}^{n-1}; |\alpha| \geq 2} w^{\alpha} a_{\alpha}, \quad w \in \mathbb{C}^{n-1},
\]

where \( a_{\alpha} \in \mathbb{C}^n \) for each \( \alpha \).

Now write \( w = [f'(r)]^\beta \hat{z} \). For notational convenience, if \( \alpha \in \mathbb{N}_0^{n-1} \) and \( t \in \mathbb{R}^{n-1} \), then write \( \alpha \cdot t = \sum_{k=1}^{n-1} \alpha_k t_k \), and let \( E(t) \) denote the diagonal operator \( \text{diag}(e^{it_1}, \ldots, e^{it_{n-1}}) \). Set \( X = [0, 2\pi]^{n-1} \subseteq \mathbb{R}^{n-1} \), and suppose that \( m \) is Lebesgue measure in \( \mathbb{R}^{n-1} \), normalized so that \( m(X) = 1 \). The Cauchy integral formula then gives, for any \( \alpha \),

\[
w^{\alpha} a_{\alpha} = \int_X e^{-i\alpha \cdot t} G(E(t)w) \, dm(t)
= \int_X e^{-i\alpha \cdot t} F(r, E(t)\hat{z}) \, dm(t) - \int_X e^{-i\alpha \cdot t} (f(r), E(t)\hat{z}) \, dm(t).
\]

The second of the two integrals in the last line is clearly equal to 0 because \( |\alpha| \geq 2 \). Since \( E(t) \) is unitary, we have

\[
|w^{\alpha}||a_{\alpha}| \leq \int_X \|F(r, E(t)\hat{z})\| \, dm(t) \leq \frac{\|z\|}{(1 - \|z\|^2)^2} \leq \frac{2(1 + r)}{(1 - r)^2},
\]

where we used the bound \( \|F(z)\| \leq \|z\|/(1 - \|z\|^2) \), true of all \( F \in \mathcal{S}_n \) [4, Corollary 8.3.9]. But now we observe that

\[
|w^{\alpha}| = \prod_{k=2}^{n} \left( [f'(r)]^\beta \rho_k \sqrt{1 - r^2} \right)^{\alpha_k - 1} = \left( \frac{1 + r}{(1 - r)^3} \right)^{|\alpha|/2} (1 - r^2)^{|\alpha|/2} \prod_{k=2}^{n} \rho_k^{\alpha_k - 1}.
\]

Where therefore solve to find

\[
||a_{\alpha}| \prod_{k=2}^{n} \rho_k^{\alpha_k - 1} \leq 2(1 - r)^{(6\beta - 1)|\alpha|/2 - 2} (1 + r)^{1 - (2\beta + 1)|\alpha|/2}.
\]

If \( |\alpha| > 4/(6\beta - 1) \), then the limit of the right-hand side as \( r \to 1^+ \) is 0. Since the left-hand side is constant, this implies that \( a_{\alpha} = 0 \) in this case.

\[
\square
\]

5 Further observations concerning \( \Phi_{G,\beta} \)

In this section, we make a pair of observations about the extension operators \( \Phi_{G,\beta} \). These observations do not require that \( \Phi_{G,\beta} \) be Loewner chain preserving, and so the constraints governing the size and degree of the terms of \( G \) discussed in Sections 3 and 4 do not apply.

Recall that, for \( n \in \mathbb{N} \), \( F \in H(B, \mathbb{C}^n) \) is a Bloch mapping provided that

\[
\sup_{z \in B} (1 - \|z\|^2)\|DF(z)\| < \infty.
\]
(See [4, Section 9.1].) Write $\mathcal{B}_1$ for the family of Bloch mappings $f$ of $\Delta$, normalized such that
\[
\sup_{z \in \Delta} (1 - |\zeta|^2)|f'(\zeta)| = f'(0) = 1.
\]
We prove the following theorem.

**Theorem 5.1.** Let $f \in S_1 \cap \mathcal{B}_1$, $\beta \in [0,1/2]$, and $G \in H(\mathbb{C}^{n-1}, \mathbb{C})$ satisfy $G(0) = 0$ and $DG(0) = 0$. Then $\Phi_{G,\beta}(f)$ is a Bloch mapping of $B$.

The case where $G = 0$ was proved by Graham, G. Kohr, and M. Kohr [6].

**Proof.** Let $z \in B$, and write $G = \sum_{j=2}^{\infty} P_j$ with $P_j \in \mathcal{P}_j(n-1)$ for all $j = 2, 3, \ldots$. For a given $u \in \partial B$, we have
\[
DF(z)u = \left( f'(z_1)u_1 + \sum_{j=2}^{\infty} [f'(z_1)]^{\beta_j} \left( DP_j(z)\hat{u} + \beta_j f''(z_1)P_j(z)u_1 \right) \right),
\]
\[
[|f'(z_1)|^{\beta} \left( \hat{u} + \beta u_1 \frac{f''(z_1)}{f'(z_1)} \hat{\zeta} \right)].
\]

We use that $|P_j(\hat{z})| \leq \|P_j\|\|\hat{z}\|^{2}$ and $|DP_j(\hat{z})\hat{u}| \leq j\|P_j\|\|\hat{z}\|^{j-1}\|\hat{u}\|$ to see that
\[
\|DF(z)\| = \sup_{u \in \partial B} \|DF(z)u\|
\]
\[
\leq |f'(z_1)| + \left( \beta \left| \frac{f''(z_1)}{f'(z_1)} \right|\|\hat{z}\| + 1 \right) \left( |f'(z_1)|^{\beta} + \sum_{j=2}^{\infty} |f'(z_1)|^{\beta_j} \|P_j\|\|\hat{z}\|^{j-1} \right).
\]

It follows from the inequality (see Lemma 3.1)
\[
\left| \frac{1 - |z_1|^2}{2} \frac{f''(z_1)}{f'(z_1)} - \Sigma \right| \leq 2
\]
that
\[
(1 - |z_1|^2) \left| \frac{f''(z_1)}{f'(z_1)} \right| \leq 6.
\]
Combining this with the inequalities $\|\hat{z}\| \leq \sqrt{1 - |z_1|^2}$ and $|f'(z_1)| \leq 1/(1 - |z_1|^2)$, we have
\[
(1 - \|z\|^2)\|DF(z)\|
\]
\[
\leq (1 - |z_1|^2)\|DF(z)\|
\]
\[
\leq 1 + (6\beta + \sqrt{1 - |z_1|^2}) \left( (1 - |z_1|^2)^{1/2 - \beta} + \sum_{j=2}^{\infty} j(1 - |z_1|^2)^{(1/2 - \beta)j} \|P_j\| \right)
\]
\[
\leq 6\beta + 2 + (6\beta + 1) \sum_{j=2}^{\infty} j\|P_j\|.
\]
The proof is therefore complete with the observation that \( \sum_{j=2}^{\infty} j \| P_j \| < \infty \). To see this, expand \( G \), using multi-indices, as

\[
G(w) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \geq 2} a_\alpha w^\alpha, \quad w \in \mathbb{C}^{n-1},
\]

where \( a_\alpha \in \mathbb{C} \) for each \( \alpha \). This series converges absolutely for all \( w \in \mathbb{C}^{n-1} \). Therefore, choosing \( w = (2, \ldots, 2) \), we have \( \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| \geq 2} 2^{|\alpha|} |a_\alpha| < \infty \). Now for any \( j = 2, 3, \ldots , \)

\[
j \| P_j \| \leq j \sup_{u \in \partial B_{n-1}} |P_j(u)| \leq j \sup_{u \in \partial B_{n-1}} \left| \sum_{|\alpha| = j} a_\alpha u^\alpha \right| \leq 2^j \sum_{|\alpha| = j} |a_\alpha|.
\]

Summing the above inequalities over \( j = 2, 3, \ldots \) gives the desired result. \( \square \)

Recall that, for \( n \in \mathbb{N} \), if \( \mathcal{F} \subseteq S_n \), then the radius of starlikeness of \( \mathcal{F} \) is the quantity

\[
r^*(\mathcal{F}) = \sup \{ \rho \in [0, 1] : F(\rho B) \text{ is starlike for all } F \in \mathcal{F} \}.
\]

It is well-known that \( r^*(S_1) = \tanh \pi/4 \). We can prove the following.

**Theorem 5.2.** Let \( \beta \in [0, 1/2] \) and \( G \in H(\mathbb{C}^{n-1}, \mathbb{C}) \) have homogeneous expansion \( G = \sum_{j=2}^{\infty} P_j \), where \( P_j = 0 \) for all \( j < 4/(2\beta + 1) \) and, if \( \beta > 1/6 \), \( P_j = 0 \) for all \( j > 4/(6\beta - 1) \). Set

\[
\alpha = \sup \left\{ \sigma \geq 0 : \sum_{j=2}^{[1/\beta]} (j-1)2^{(1-6\beta)j/2} \sigma^{j-1} \| P_j \| + \sum_{j=1/\beta+1}^{\infty} (j-1)2^{(2\beta+1)j/2} \sigma^{j-1} \| P_j \| \leq C_\beta \right\},
\]

where \( C_\beta \) is as in Theorem 3.2. If \( \alpha < \tanh \pi/4 \), then \( r^*(\Phi_{G,\beta}(S_1)) \geq \alpha \). If \( \alpha \geq \tanh \pi/4 \), then \( r^*(\Phi_{G,\beta}(S_1)) = \tanh \pi/4 \).

**Proof.** Let \( \rho = \min \{ \alpha, \tanh \pi/4 \} \), and let \( f \in S_1 \). Then \( f \) is starlike on \( \rho \Delta \), and we define \( g \in S^*_n \) by \( g(\zeta) = f(\rho \zeta)/\rho \) for \( \zeta \in \Delta \). Let \( H = \sum_{j=2}^{\infty} \rho^{j-1} P_j \). A direct calculation shows that

\[
\frac{1}{\rho} [\Phi_{G,\beta}(f)](\rho z) = [\Phi_{H,\beta}(g)](z), \quad z \in B.
\]

Since \( \rho \leq \alpha \), Corollary 3.5 implies that \( \Phi_{H,\beta}(g) \in S^*_n \), showing that \( \Phi_{G,\beta}(f) \) is starlike on \( \rho B \). Hence \( r^*(\Phi_{G,\beta}(S_1)) \geq \rho \).

In the case that \( \alpha \geq \tanh \pi/4 \), let \( \sigma \in (\tanh \pi/4, 1) \). Choose \( f \in S_1 \) such that \( f \) is not starlike on \( \sigma \Delta \). Observe that

\[
[\Phi_{G,\beta}(f)](\sigma B) \cap \text{span}\{ e_1 \} = \{ f(\zeta)e_1 : \zeta \in \sigma \Delta \}
\]

is not a starlike set. It follows that \( [\Phi_{G,\beta}(f)](\sigma B) \) is not a starlike set. We conclude that, in this case, \( r^*(\Phi_{G,\beta}(S_1)) = \tanh \pi/4 \). \( \square \)
Observe that if $G$ satisfies (3.3), then $\alpha \geq 1 > \tanh \pi/4$, and therefore $r^*(\Phi_{G,\beta}(S_1)) = \tanh \pi/4$. This gives further support to the conjecture (see [4]) that $r^*(S_0^n) = \tanh \pi/4$.

In the case that $\beta = 1/m$ and $G = P \in \mathcal{P}_m(n - 1)$, we can use the bounds in Corollary 3.3 to improve the results in Theorem 5.2. The method to do so is clear from the proof of Theorem 5.2.

6 Remarks on extreme and support points for Loewner chain preserving extension operators

In some recent work, Graham, Kohr, and Pfaltzgraff [7] study extreme points and support points of the family $\Phi_{0,1/2}(S_1)$. (Recall that $\Phi_{0,1/2}$ is the Roper–Suffridge extension operator.) Some of their observations remain valid if $\Phi_{0,1/2}$ is replaced by any Loewner chain preserving extension operator, as we show in the following.

Recall that if $X$ is a locally convex topological vector space and $A \subseteq X$, then a vector $x \in A$ is an extreme point of $A$ if, whenever $y, z \in A$ satisfy $1 - \lambda y + \lambda z = x$ for some $\lambda \in (0, 1)$, it follows that $y = z = x$. A vector $x \in A$ is a support point of $A$ if there exists a continuous linear functional $\ell : X \to \mathbb{C}$ such that

$$\text{Re} \, \ell(x) = \max_{y \in A} \text{Re} \, \ell(y).$$

We denote the set of extreme points of $A$ by $\text{ex} \, A$ and the set of support points of $A$ by $\text{supp} \, A$. In the following, our locally convex topological vector spaces are understood to be $H(\Delta, \mathbb{C})$ and $H(B, \mathbb{C}^n)$.

In the following result, observe that $\Phi$ is not required to be a Loewner chain preserving extension operator.

**Theorem 6.1.** Let $\Phi : \mathcal{L}S_1 \to \mathcal{L}S_n$ be an extension operator, and let $\mathcal{F} \subseteq \mathcal{L}S_1$. Then $\Phi(\text{ex} \, \mathcal{F}) \subseteq \text{ex} \, \Phi(\mathcal{F})$ and $\Phi(\text{supp} \, \mathcal{F}) \subseteq \text{supp} \, \Phi(\mathcal{F})$.

**Proof.** Let $f \in \text{ex} \, \mathcal{F}$. Suppose that there are $g, h \in \mathcal{F}$ and $\lambda \in (0, 1)$ such that $\Phi(f) = (1 - \lambda)\Phi(g) + \lambda\Phi(h)$. Evaluating these functions at $\zeta e_1$ for any $\zeta \in \Delta$ gives

$$f(\zeta) = (1 - \lambda)g(\zeta) + \lambda h(\zeta).$$

It follows that $g = h = f$, and hence $\Phi(g) = \Phi(h) = \Phi(f)$. Thus $\Phi(f) \in \text{ex} \, \Phi(\mathcal{F})$.

Now let $f \in \text{supp} \, \mathcal{F}$. Then there is a continuous linear functional $\ell : H(\Delta, \mathbb{C}) \to \mathbb{C}$ such that $\text{Re} \, \ell(f) = \max_{g \in \mathcal{F}} \text{Re} \, \ell(g)$. For any $F \in H(B, \mathbb{C}^n)$, define the function $\tilde{F} \in H(\Delta, \mathbb{C})$ by

$$\tilde{F}(\zeta) = \langle F(\zeta e_1), e_1 \rangle, \quad \zeta \in \Delta.$$ Define $L : H(B, \mathbb{C}^n) \to \mathbb{C}$ by $L(F) = \ell(\tilde{F})$. It is easy to see that $L$ is a continuous linear functional and that for all $f \in \mathcal{L}S_1$, $L(\Phi(f)) = \ell(f)$. Therefore

$$\max_{G \in \Phi(\mathcal{F})} \text{Re} \, L(G) = \max_{g \in \mathcal{F}} \text{Re} \, \ell(g) = \text{Re} \, \ell(f) = \text{Re} \, L(\Phi(f)),$$

showing that $\Phi(f) \in \text{supp} \, \Phi(\mathcal{F})$. \hfill \qed
Example 6.2. Consider Theorem 6.1 in the case where $\mathcal{F} = \mathcal{K}_1$ and $\Phi = \Phi_{Q,1/2}$ for $Q \in \mathcal{P}_2(n-1)$ satisfying $\|Q\| \leq 1/2$. For each $\theta \in \mathbb{R}$, let $\varphi_\theta$ be given by (2.6). Then the mappings $\Phi_{Q,1/2}(\varphi_\theta)$ are extreme points of $\Phi_{Q,1/2}(\mathcal{K}_1) \subseteq \mathcal{K}_n$. In [10], it is proved that these mappings are actually extreme points of $\mathcal{K}_n$ if and only if $Q$ has the form

$$Q(w) = \frac{1}{2} \sum_{k=1}^{n-1} \langle w, v_k \rangle^2, \quad w \in \mathbb{C}^{n-1},$$

where $\{v_1, \ldots, v_{n-1}\}$ is an orthonormal basis of $\mathbb{C}^{n-1}$.

Pell [13] proved that the sets $\text{ex}\mathcal{S}_1$ and $\text{supp}\mathcal{S}_1$ are preserved under the Loewner variation. In other words, if $f \in \text{ex}\mathcal{S}_1$ (respectively $f \in \text{supp}\mathcal{S}_1$), then $e^{-t}f(\cdot, t) \in \text{ex}\mathcal{S}_1$ (respectively $e^{-t}f(\cdot, t) \in \text{supp}\mathcal{S}_1$) for all $t \geq 0$, where $f: \Delta \times [0, \infty) \to \mathbb{C}$ is a Loewner chain such that $f(\zeta) = f(\zeta, 0)$ for $\zeta \in \Delta$.

The following result is immediate.

Corollary 6.3. Let $\Phi: \mathcal{L}\mathcal{S}_1 \to \mathcal{L}\mathcal{S}_n$ be a Loewner chain preserving extension operator, let $f \in \mathcal{S}_1$ be the first element of a Loewner chain $f: \Delta \times [0, \infty) \to \mathbb{C}$, and let $F$ be the corresponding Loewner chain given by (2.1). If $f \in \text{ex}\mathcal{S}_1$, then $e^{-t}F(\cdot, t) \in \text{ex}\Phi(\mathcal{S}_1)$ for all $t \geq 0$. If $f \in \text{supp}\mathcal{S}_1$, then $e^{-t}F(\cdot, t) \in \text{supp}\Phi(\mathcal{S}_1)$ for all $t \geq 0$.

Proof. This follows from Theorem 6.1, the results of Pell, and that $e^{-t}F(\cdot, t) = \Phi(e^{-t}f(\cdot, t))$. □

References


